Topology (Math 3281)

Solutions to Problem Set 3 21.11.14

1. Let X be a Hausdorff space. We need to show that $X \times X - \Delta(X)$ is open. Let $(x, y) \in X \times X - \Delta(X)$. This means that $x \neq y$. Since X is Hausdorff, we get open neighborhoods U, V with $U \cap V = \emptyset$. Then $U \times V \subset X \times X - \Delta(X)$ is an open set containing (x, y). It follows that $X \times X - \Delta(X)$ is open, so $\Delta(X)$ is closed.

Now assume that $\Delta(X) \subset X \times X$ is closed. Let $x \neq y \in X$. As $X \times X - \Delta(X)$ is open and contains (x, y), we can find open sets $U, V \subset X$ such that $U \times V \subset X \times X - \Delta(X)$ is a neighborhood of (x, y). But this means that $U \cap V = \emptyset$ and X is Hausdorff.

2. For every integer $n \geq 1$ notice that the interval $(1/n - \varepsilon, 1/n + \varepsilon)$ does not contain other points of X than $1/n$, provided that $\varepsilon < 1/(n(n + 1))$. Hence $f_n : X \to \{0, 1\}$ given by

$$f_n(x) = \begin{cases} 0 & x = 1/n \\ 1 & x \neq 1/n \end{cases}$$

is continuous. Hence any set $A \subset X$ which contains $1/n$ and at least one more point of X admits a continuous surjection to a discrete space with more than one point. So the only connected subset of X containing $1/n$ is $\{1/n\}$. This shows that each $\{1/n\}$ is a component. Now $\{0\}$ is the component containing 0, as intersections of different components are empty, and every point different from 0 is its own component.

3. Let $f : X \to Y$ be a continuous map with X path-connected. We need to show that $f(X)$ is path-connected. Let $a, b \in f(X)$. Then there exist $x, y \in X$ with $f(x) = a$ and $f(y) = b$. Since X is path-connected, there is a map $\gamma : [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Now $f \circ \gamma : [0, 1] \to f(X)$ is the required path from a to b.

4. The spaces in (b) and (e) are homeomorphic, as straightening of the edges in (e) shows. The rest is all differnet up to homeomorphism. To see that, note that (a) contains a point so that removing it leaves us with four components. Noe of the other spaces have that property. Removing the middle point in (b) gives three components. Removing a single point in (c) or (d) keeps the space connected. To see that (c) and (d) are not homeomorphic, note that removing two points from (c) always gives a disconnected space,

but by carefully choosing we can remove two points from (d) so that the result is still connected.