Topology (Math 3281)

Homework Problem Set 4 21.11.14

This set of homeworks will be collected in the lecture on 05.12.14.

1. Let $Y = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^5 + z_2^2 + z_3^2 = 0\}$, and $X = Y \cap S^5 \subset \mathbb{C}^3$. Decide whether X is compact. Note that we can identify $\mathbb{C}^3$ with $\mathbb{R}^6$ to give it the standard topology.

2. Let X, Y be a topological spaces, $A, B \subset X$ be two closed sets such that $A \cup B = X$. Let $f : X \to Y$ be a function such that the restriction of f to both A and B are continuous. Show that f is continuous.

3. Let X be a topological space, $Y, Z \subset X$ closed subsets with $Y \cup Z$ connected and $Y \cap Z$ connected. Show that both Z and Y are connected. Hint: Try to extend a map $f : Z \to \{0, 1\}$ to a map $F : Z \cup Y \to \{0, 1\}$.

4. On 13th of November Alexander Grothendieck, arguably the greatest mathematician of the 20th century, died. To honour his achievements, we have the following memorial question.

   Let R be a commutative ring and

   $$Spec(R) = \{P \subset R \mid P \text{ is a prime ideal in } R\}.$$

   In words, $Spec(R)$ is the set of prime ideals in R. If $I \subset R$ is an ideal, let

   $$Z(I) = \{P \in Spec(R) \mid I \subset P\} \subset Spec(R).$$

   (a) Show that if $I \subset J$ are ideals, then $Z(J) \subset Z(I)$.

   (b) Show that $Z(I \cdot J) = Z(I) \cup Z(J)$ for I, J ideals.

   (c) Show that $\tau = \{Spec(R) - Z(I) \mid I \subset R \text{ an ideal}\}$ defines a topology on $Spec(R)$. Note that this means the $Z(I)$ are the closed sets.

   (d) Show that $Spec(\mathbb{Z})$ is compact.