TOPOLOGY III

DIRK SCHÜTZ

1. Metric Spaces and continuous functions

Definition 1.1. A *metric space* is a pair (M, d) where M is a set and $d\colon M \times M \to [0, \infty)$ is a function satisfying the following:

(1) $d(x, y) = 0$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all all $x, y \in M$.
(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

We call d a *metric* on M then.

Example 1.2.

(1) $\mathbb{R}^n$ with the metric

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

or the metric

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$$

or the metric

$$d_\infty(x, y) = \max\{|x_i - y_i| \mid i = 1, \ldots, n\}.$$

(2) Let $C([a, b]) = \{f\colon [a, b] \to \mathbb{R} \mid f \text{ is continuous}\}$. Then

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [a, b]\}$$

is a metric on $C([a, b])$.

(3) Let M be any set. Then

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric, called the *discrete metric on* X.

Definition 1.3. Let (M, d) be a metric space.

(1) A *sequence in* M is a function $a\colon \mathbb{N} = \{0, 1, 2, 3, \ldots\} \to M$. We write $(a_n)_{n \in \mathbb{N}}$ for the sequence, where $a_n = a(n)$.

Date: December 12, 2014.

(2) Let $a \in M$. Then the sequence $(a_n)_{n\in\mathbb{N}}$ *converges to* a, if for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $d(a, a_n) < \varepsilon$ for all $n \geq n_0$. We write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$.

Definition 1.4. Let (M, d_M) and (N, d_N) be metric spaces, and $a \in M$. A function $f\colon M \to N$ is called *continuous* at $a \in M$, if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in M$ with $d_M(x, a) < \delta$ we have $d_N(f(x), f(a)) < \varepsilon$. The function is called *continuous*, if it is continuous for every $a \in M$.

Example 1.5.

(1) Any continuous function $f\colon \mathbb{R} \to \mathbb{R}$ that you have seen in Calculus.
(2) Define $F\colon C[0, 1] \to C[0, 1]$ by

$$F(f)(x) = \int_0^x f(t)dt$$

This is continuous with the metric from Example 1.2 (2).

Lemma 1.6. *Let M, N be metric spaces, $f\colon M \to N$ a function and $a \in M$. Then f is continuous at a if and only if for every sequence $(a_n)_{n\in\mathbb{N}}$ in M with $a_n \to a$ we have $f(a_n) \to f(a)$.*

Example 1.7. Let X be a set with the discrete metric, and let $f\colon X \to \mathbb{R}$ be a function. Then f is continuous. On the other hand, $\mathrm{id}\colon (\mathbb{R}, d_2) \to (\mathbb{R}, d)$ with d the discrete metric on $\mathbb{R}$ is not continuous.

Definition 1.8. Let (M, d) be a metric space, $r > 0$ and $x \in M$. Then

$$B(x; r) = \{y \in M \mid d(x, y) < r\}$$

is called the *open ball of radius r around $x \in M$*, and

$$D(x; r) = \{y \in M \mid d(x, y) \leq r\}$$

is called the *closed ball of radius r around $x \in M$*.

Example 1.9. Let $M = \mathbb{R}^2$ with d_2 the euclidean metric. Then $B(0; 1)$ is a round ball. With the metric d_1 it is shaped like a diamond and with d_∞ it is shaped like a square.
With the discrete metric $B(x; 1) = \{x\}$, while $D(x; 1)$ is the whole space.

Definition 1.10. Let M be a metric space. A subset $U \subset M$ is called *open*, if for every $x \in U$ there is $\varepsilon > 0$ such that $B(x; \varepsilon) \subset U$. A subset $A \subset M$ is called *closed*, if $M - A$ is open.

Lemma 1.11. *Let M be a metric space.*

(1) *$B(x; r)$ is open for all $x \in M$, $r > 0$.*
(2) *$D(x; r)$ is closed for all $x \in M$, $r > 0$.*
(3) *M and $\emptyset$ are both open and closed.*
(4) *an arbitrary union of open sets is open.*

(5) *a finite intersection of open sets is open.*
(6) *a finite union of closed sets is closed.*
(7) *an arbitrary intersection of closed sets is closed.*

Example 1.12. Let $M = \mathbb{R}^2$ with the euclidean metric. Then $\{0\}$ is closed, but not open. Also,

$$\{0\} = \bigcap_{i=1}^{\infty} B(0; \frac{1}{i})$$

so the arbitrary intersection of open sets need not be open.

$D(0;1)$ is closed, but not open.

If (X,d) is discrete, then any $A \subset X$ is open and closed.

Proposition 1.13. *Let M, N be metric spaces and $f\colon M \to N$ a function.*

(1) *Let $a \in M$. Then f is continuous at a if and only if for every open subset $V \subset N$ with $f(x) \in V$ there is an open subset $U \subset M$ with $a \in M$ and $f(U) \subset V$.*
(2) *f is continuous if and only if $f^{-1}(U)$ is open in M for every open $U \subset N$.*

2. Topological Spaces

Definition 2.1. Let X be a set. A *topology* on X is a subset $\tau \subset \wp(X) = \{A \subset X\}$ which satisfy the following:

(1) $\emptyset$ and X are in τ.
(2) If $U_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$.
(3) If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

The pair (X, τ) is called a *topological space*. The elements of τ are called then *open subsets* of X.

Example 2.2.

(1) Let (M,d) be a metric space, and τ_d the collection of open sets in the sense of Definition 1.10. Then (M, τ_d) is a topological space.
(2) Let X be any set, and $\tau = \wp(X)$. This is called the *discrete* topology.
(3) Let X be any set, and $\tau = \{\emptyset, X\}$. This is called the *indiscrete* topology.
(4) Let $X = \{0,1,2,3,\ldots\}$ and $\tau = \{\emptyset\} \cup \{U \subset X \mid X - U \text{ is finite}\}$.
(5) Let (X,τ) be a topological space, $A \subset X$. Then

$$\tau_A = \{A \cap U \mid U \in \tau\}$$

is a topology, called the *induced*, or *subspace* topology. Note that $\mathbb{Z} \subset \mathbb{R}$ is discrete in the subspace topology, and $\mathbb{Q} \subset \mathbb{R}$ is not discrete in the subspace topology.
(6) On $\mathbb{R}^n$ we have metrics d_1, d_2, d_∞. They all induce the same topology on $\mathbb{R}^n$. This is called the *standard* topology on $\mathbb{R}^n$.

Definition 2.3. Let (X, τ) be a topological space. A subset $A \subset X$ is called *closed*, if its complement $X - A$ is open.

Example 2.4. If X is discrete, every $A \subset X$ is open and closed.

Lemma 2.5. *Let X be a topological space. Then X and $\emptyset$ are closed. Furthermore, any intersection of closed sets is closed, and finite unions of closed sets are closed.*

Definition 2.6. A topological space X is called *Hausdorff*, if whenever $x, y \in X$ with $x \neq y$, there exist open sets $U, V \subset X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Lemma 2.7. *Let M be a metric space. Then M is a Hausdorff space.*

Example 2.8. Let X be a set with more than one element, and give it the indiscrete topology. Then X is not a Hausdorff space.

Definition 2.9. Let X, Y be topological spaces. A function $f\colon X \to Y$ is called *continuous*, if for every open subset $U \subset Y$ the inverse image $f^{-1}(U)$ is open in X. Let $x \in X$. The function f is called *continuous at x*, if for every open set $U \subset Y$ with $f(x) \in U$ there is an open set $V \subset X$ with $x \in V$ such that $f(V) \subset U$. A continuous function is also called a *map*.

Lemma 2.10. *Composition of continuous functions is a continuous function.*

Example 2.11. Let X be a topological space and $A \subset X$. The inclusion $i\colon A \to X$ is continuous, if A is given the subspace topology.

Lemma 2.12. *Let $f\colon X \to Y$ be a function between topological spaces X and Y. Then f is continuous if and only if $f^{-1}(A)$ is closed for every $A \subset Y$ closed.*

Example 2.13. The n-sphere is defined as

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

and is a closed subset of $\mathbb{R}^{n+1}$.
Denote by $M_{n,n}(\mathbb{R})$ the set of $n \times n$-matrices, topologized as $\mathbb{R}^{n^2}$. Then $\mathrm{GL}_n(\mathbb{R})$, the set of invertible $n \times n$-matrices is an open subset of $M_{n,n}(\mathbb{R})$. Also, $\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) \,|\, \det A = 1\}$ is a closed subset of $M_{n,n}(\mathbb{R})$.
The *orthogonal group* is defined as

$$\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) \,|\, AA^t = I\}$$

which is also a closed subset. The *special orthogonal group* is

$$\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}_n(\mathbb{R})$$

and it is closed as an intersection of closed sets.

Definition 2.14. Let X, Y be topological spaces. A map $h\colon X \to Y$ is called a *homeomorphism*, if h is bijective and h^{-1} is continuous. In that case X and Y are called *homeomorphic*, and we write $X \approx Y$. Note that h induces a bijection between τ_X and τ_Y.

Example 2.15. The interval $(0, 1)$ is homeomorphic to $\mathbb{R}$. We will see later that $[0, 1)$ is not homeomorphic to $\mathbb{R}$.

$\mathbb{R}$ with the standard topology is not homeomorphic to $\mathbb{R}$ with the discrete topology.

3. Interiors and Closures, limit points and product spaces

Definition 3.1. Let X be a topological space and $x \in X$, $A \subset X$.

(1) A *neighborhood of* x is a set N which contains an open set $U \subset X$ with $x \in U$.
(2) A point $x \in X$ is called a *limit point of* A*, if every neighborhood* N *of* x *satisfies* $N - \{x\} \cap A \neq \emptyset$.

Example 3.2.

(1) $X = \mathbb{R}$, $0 \in X$, $(-\frac{1}{2}, \frac{1}{2})$ is a neighborhood of 0.
(2) If $U \subset X$ is open, then U is a neighborhood for every $x \in U$.
(3) Let $A = \{\frac{1}{n} \in \mathbb{R} \,|\, n \in \mathbb{Z} - \{0\}\}$. Then A has exactly one limit point 0, and $0 \notin A$.

Lemma 3.3. *Let* X *be a Hausdorff space,* $A \subset X$ *and* $x \in X$ *a point which has a neighborhood* N *with* $N - \{x\} \cap A$ *finite. Then* x *is not a limit point of* A.

Definition 3.4. Let X be a topological space, $A \subset X$. The *interior of* A, denoted A°, is the largest open set contained in A. The *closure of* A, denoted $\bar{A}$, is the smallest closed set which contains A. More precisely,

$$A^\circ = \bigcup_{U \subset A \text{ open}} U \qquad \bar{A} = \bigcap_{C \supset A \text{ closed}} C$$

A subset $A \subset X$ is called *dense*, if $\bar{A} = X$.

Example 3.5. Let $\mathbb{Q} \subset \mathbb{R}$, then $\mathbb{Q}^\circ = \emptyset$ and $\bar{\mathbb{Q}} = \mathbb{R}$.

Lemma 3.6. *Let* $A \subset X$. *Then* $\bar{A} = A \cup$ *limit points of* A.

Lemma 3.7. *Let* M *be a metric space,* $A \subset M$ *and* x *a limit point of* A. *Then there exists a sequence* $x_n \in A$ *such that* $\lim x_n = x$. *Furthermore, if* $x \in M - A$ *and there is a sequence* $x_n \in A$ *with* $\lim x_n = x$, *then* x *is a limit point of* A.

Definition 3.8. Let X be a topological space. A *basis* $\mathcal{B}$ *for the topology of* X is a collection of open sets such that every open set U is the union of elements of $\mathcal{B}$.

Example 3.9.

(1) Let M be a metric space. The collection $\{B(x;r)\,|\,x \in M, R > 0\}$ is a basis for the topology of M.
(2) Let $X = \mathbb{R}^n$ with the standard topology. The collection $\{B(q;\frac{1}{m})\,|\,q \in \mathbb{Q}^n, m \geq 1 \text{ integer}\}$ is a countable basis of the topology.

Theorem 3.10. *Let $f\colon X \to Y$ be a function between topological spaces. The following are equivalent.*

(1) *f is continuous.*
(2) *If $\mathcal{B}$ is a basis for the topology of Y, then $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.*
(3) *$f(\bar{A}) \subset \overline{f(A)}$ for every subset $A \subset X$.*
(4) *$\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for every subset $B \subset Y$.*
(5) *$f^{-1}(B)$ is closed for every closed set $B \subset Y$.*

Remark 3.11. A basis for a topology usually gives a collection of open sets which are somewhat easy to handle. By 3.10(2) we see that continuity is captured by a basis. Another situation is that one may have a set X and a collection $\mathcal{B}$ of subsets that one would like to be open. Then one would like a topology for which $\mathcal{B}$ is a basis.

Theorem 3.12. *Let X be a set and $\mathcal{B}$ a collection of subsets of X such that*

(1) *For each $x \in X$ there is a $B \in \mathcal{B}$ with $x \in B$.*
(2) *If $x \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}$, then there is a $B_3 \in \mathcal{B}$ with $x \in B_3 \subset B_1 \cap B_2$.*

Then there is a unique topology $\tau_{\mathcal{B}}$ on X of which $\mathcal{B}$ is a basis.

Definition 3.13. Let X, Y be topological spaces. The *cartesian product* of X and Y is the set

$$X \times Y = \{(x,y)\,|\,x \in X, y \in Y\}$$

and it is given the *product topology*, obtained through Theorem 3.12 by using

$$\mathcal{B}_{X\times Y} = \{U \times V\,|\,U \subset X, V \subset Y\}$$

Example 3.14.

(1) Let $X = \mathbb{R}$, $Y = \mathbb{R}$. Then $X \times Y = \mathbb{R}^2$, and the product topology agrees with the standard topology.
(2) Let $X = S^1 = Y$. Then $T^2 = S^1 \times S^1$ is called the torus. We can also form the n-torus by defining

$$T^n = S^1 \times T^{n-1}.$$

Proposition 3.15. *Let X, Y be topological spaces, and $X \times Y$ given the product topology. The projections $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are both continuous and map open sets to open sets. Furthermore, the product topology is the smallest topology so that both projections are continuous.*

Proposition 3.16. *Let X, Y, Z be topological spaces. A function $f\colon Z \to X \times Y$ is continuous if and only if both $p_X \circ f\colon Z \to X$ and $p_Y \circ f\colon Z \to Y$ are continuous.*

Example 3.17. Let $f\colon X \to \mathbb{R}^n$ be a function. We then have n coordinate functions $f_1, \dots, f_n\colon X \to \mathbb{R}$, and f is continuous if and only if these coordinate functions are continuous.

Proposition 3.18. *Let X, Y be non-empty topological spaces. Then $X \times Y$ is a Hausdorff space if and only if X and Y are Hausdorff spaces.*

4. Connectedness

Definition 4.1. A topological space is connected if whenever it is decomposed as a union $X = A \cup B$ of two non-empty sets A and B, then $\bar{A} \cap B \neq \emptyset$ or $A \cup \bar{B} \neq \emptyset$.

Example 4.2.

(1) $\mathbb{R}$ is connected.
(2) $\mathbb{Q}$ is not connected.
(3) The connected subsets of $\mathbb{R}$ are the intervals.
(4) Let $X = \{0, 1\}$ with the discrete topology. Then X is not connected.
(5) If the topology of X is given by $\tau = \{\emptyset, \{1\}, \{0, 1\}\}$, then X is connected.

Theorem 4.3. *Let X be a topological space. The following are equivalent.*

(1) *X is connected.*
(2) *The only subsets which are open and closed are X and the emptyset.*
(3) *X cannot be written as the union of two disjoint, non-empty open sets.*
(4) *There is no continuous surjective function from X to a discrete space with more than one point.*

Theorem 4.4. *The continuous image of a connected space is connected. In particular, if $h\colon X \to Y$ is a homeomorphism with X connected, then Y is connected.*

Example 4.5.

(1) $\det\colon GL_n(\mathbb{R}) \to \mathbb{R} - \{0\}$ is onto. Hence $GL_n(\mathbb{R})$ is not connected.
(2) $O(n)$ is not connected.
(3) $X = (0, 1)$ and $Y = (0, 1]$ are not homeomorphic.

Proposition 4.6. *Let X be a topological space and $Z \subset X$. If Z is connected and $\bar{Z} = X$, then X is connected.*

Corollary 4.7. *Let X be a topological space, $Z \subset X$ connected and $Y \subset X$ with $Z \subset Y \subset \bar{Z}$. Then Y is connected. In particular, the closure of a connected set is connected.*

Proposition 4.8. *Let $\mathcal{A} = \{A_i \,|\, i \in I\}$ be a collection of subsets of X such that $\bigcup_{i\in I} A_i = X$. Assume that each A_i is connected and for each $i, j \in I$ we have $\bar{A}_i \cap \bar{A}_j \neq \emptyset$. Then X is connected.*

Theorem 4.9. *If X and Y are connected, then $X \times Y$ is connected.*

Example 4.10.

(1) $\mathbb{R}^n$ is connected.
(2) B^n is connected.
(3) D^n is connected.
(4) S^n is connected for $n \geq 1$.
(5) T^n is connected for $n \geq 1$.

Definition 4.11. A *component* of a topological space X is a maximal connected subset of X.

Proposition 4.12. *Each component of a topological space is a closed subset and if C_1 and C_2 are different components, then $C_1 \cap C_2 = \emptyset$. Also, a topological space is the union of its components.*

Example 4.13.

(1) If X is connected, there is only one component.
(2) If X is discrete, each point is a component.
(3) In $\mathbb{Q}$ every point is a component.

Definition 4.14. A *path* in a topological space X is a continuous function $\gamma\colon [0,1] \to X$, we say γ is a path from $\gamma(0)$ to $\gamma(1)$.
A topological space is called *path connected*, if for any two points $x, y \in X$ there is a path from x to y.

Proposition 4.15. *A path connected space is connected.*

Example 4.16. Let

$$Z = \{(x, \sin \frac{\pi}{x}) \in \mathbb{R}^2 \,|\, 0 < x \leq 1\}.$$

Then $\bar{Z}$ is connected, but not path connected.

5. Compactness

Definition 5.1. A topological space is *compact*, if every open cover of X admits a finite subcover. An open cover of X is a collection of open sets $U_i \subset X$ such that $\bigcup_{i\in I} U_i = X$.

Example 5.2.

(1) If X is a finite set, then X is compact.
(2) $\mathbb{R}$ is not compact.
(3) Let X be an infinite set with the topology $\tau = \{U \subset X \,|\, X - U \text{ is finite.}\} \cup \{\emptyset\}$. Then X is compact.

(4) The closed interval $[0,1]$ is compact.

Proposition 5.3. *The continuous image of a compact space is compact. In particular, if $X \approx Y$ and X is compact, then Y is compact.*

Proposition 5.4. *A closed subset of a compact space is compact.*

Theorem 5.5. *A compact subset of a Hausdorff space X is closed in X.*

Corollary 5.6. *A bijective continuous function from a compact space to a Hausdorff space is a homeomorphism.*

Example 5.7. Let X be an infinite set with the topology $\tau = \{U \subset X \mid X - U \text{ is finite.}\} \cup \{\emptyset\}$. Every subset of X is compact, not just the closed ones. But X is not Hausdorff.

Theorem 5.8. *Let X, Y be compact spaces. Then $X \times Y$ is compact.*

Theorem 5.9 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Example 5.10.

(1) S^n is compact.
(2) The torus $T^n = S^1 \times \cdots \times S^1$ is compact.
(3) $X = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^3 = 1\}$ is not compact.

Remark 5.11. In a metric space the equivalence between compact sets and closed and bounded sets usually does not work.

Corollary 5.12. *Let $f\colon X \to \mathbb{R}$ be a map from a compact space to the reals. Then f attains its maximum and minimum.*

Theorem 5.13 (Bolzano-Weierstrass)**.** *An infinite subset of a compact space must have a limit point.*

Theorem 5.14 (Lebesgue's Lemma)**.** *Let X be a compact metric space and let $\mathcal{U}$ be an open cover of X. Then there exists a $\delta > 0$ (called a* Lebesgue number *of $\mathcal{U}$) such that any subset of X of diameter less than δ is contained in some member of $\mathcal{U}$.*

If $A \subset M$ with M a metric space, the diameter is defined by

$$\operatorname{diam}(A) = \sup\{d(a,b) \mid a, b \in A\}.$$

6. Quotient Spaces

Definition 6.1. Let X be a topological space, Y a set and $p\colon X \to Y$ a function which is surjective. Define a topology on Y as follows: we say $U \subset Y$ is open if and only if $p^{-1}(U)$ is open in X. This topology on Y is called the *quotient topology* on Y, and Y is called a *quotient space* of X, with $p\colon X \to Y$ a *quotient map.*

Example 6.2. Let X be a topological space, and $\sim$ an equivalence relation. For $x \in X$, the equivalence class $[x] = \{y \in X \mid x \sim y\}$, and let

$$X/\sim = \{[x] \mid x \in X$$

the set of equivalence classes.

Then we have a surjection $p\colon X \to X/\sim$ with $p(x) = [x]$ and we can give $X/\sim$ the quotient topology.

If $X = \mathbb{R}$, define $x \sim y$ if and only if $x - y \in \mathbb{Z}$. We will see that $\mathbb{R}/\sim \approx S^1$.

On $X = D^n$ define $\sim$ by $x \sim y$ if and only if $x = y$, or both $x, y \in S^{n-1}$. Then $D^n/\sim \approx S^n$, as we shall see.

Remark 6.3. An equivalence relation on X is the same as a partition $\mathcal{P}$ of X, which is a collection of disjoint subsets (the equivalence classes) which cover the whole of X. Every surjective function between sets $f\colon X \to Y$ produces a partition of X via $f^{-1}(\{y\})$, $y \in Y$.

Proposition 6.4. *Let $p\colon X \to Y$ be a quotient map, and Z a topological space. A function $f\colon Y \to Z$ is continuous if and only if $f \circ p\colon X \to Z$ is continuous.*

Definition 6.5. Let X, Y be topological spaces, and $f\colon X \to Y$ a map which is onto. Then f is called an *identification map*, if $U \subset Y$ is open if and only if $f^{-1}(U) \subset X$ is open.

Remark 6.6. Let $f\colon X \to Y$ be an identification map. Then f is an onto function and we can give Y the quotient topology. But this is the same topology on Y that it had before. An identification map is pretty much the same as a quotient map. The only difference is that in the case of a quotient map Y was a set while in the case of an identification map Y already has a topology.

If an identification map is injective, it is a homeomorphism.

Example 6.7. Let $\mathrm{id}\colon (\mathbb{R}, \tau_{disc}) \to (\mathbb{R}, \tau_{std})$ be the identity between the discrete and the standard topology. This is *not* an identification map: $U \subset \mathbb{R}$ open in the standard topology is not equivalent to U open in the discrete topology.

Proposition 6.8. *Let $f\colon X \to Y$ be an onto map. If f maps closed sets to closed sets (or open sets to open sets), then f is an identification map.*

Corollary 6.9. *Let $f\colon X \to Y$ be an onto map. If X is compact and Y Hausdorff, then f is an identification map.*

Example 6.10. Let X be a topological space, and $A \subset X$. Then define an equivalence relation $\sim$ by $x \sim y$ if and only if $x = y$ or both $x, y \in A$. Then A is one equivalence class, and all other equivalence classes consist of one point. The resulting quotient space is denoted X/A.

7. Topological Groups

Definition 7.1. A *topological group* G is a Hausdorff space G which is also a group, such that multiplication $\cdot\colon G \times G \to G$ and inversion $i\colon G \to G$ are continuous.

Example 7.2.

(1) $\mathbb{R}^n$ with addition.
(2) $\mathbb{C} - \{0\}$ with multiplication. Then $S^1 \subset \mathbb{C}$ is a subgroup.
(3) The set of invertible $n \times n$ matrices $\mathrm{GL}_n(\mathbb{R})$, with subgroups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$.
(4) Any subgroup H of a topological group G is a topological group.
(5) Let $\mathbb{H} = \mathbb{C} \times \mathbb{C}$, the set of quaternions, with topology from $\mathbb{R}^4$. We can define a multiplication by
$$(z_1, z_2) \cdot (w_1, w_2) = (z_1 w_1 - \bar{w}_2 z_2, w_2 z_1 + z_2 \bar{w}_1).$$
This turns $\mathbb{H} - \{0\}$ into a multiplicative group, and S^3 forms a subgroup.
(6) Any group G with the discrete topology is a topological group.

Remark 7.3. Let G be a topological group. For $x \in G$ define $L_x\colon G \to G$ by $L_x(g) = xg$, called *left translation* by x. This is continuous, and has an inverse, namely $L_{x^{-1}}$. In particular left translation is a homeomorphism. Similarly, right translation R_x is a homeomorphism.

Proposition 7.4. *Let G be a topological group and K the connected component of G which contains the identity element. Then K is a closed normal subgroup of G.*

Remark 7.5. We know that $\mathrm{O}(n)$ is not connected, but we will see that $\mathrm{SO}(n)$ is connected. In particular, $\mathrm{SO}(n)$ is the K in the case of $G = \mathrm{O}(n)$.

Theorem 7.6. $\mathrm{O}(n)$ *and* $\mathrm{SO}(n)$ *are compact.*

Definition 7.7. An *action* of a topological group G on a topological space X is a map $\bullet\colon G \times X \to X$ such that

(1) $(hg) \bullet x = h \bullet (g \bullet x)$ for all $h, g \in G$ and $x \in X$.
(2) $1 \bullet x = x$ for all $x \in X$.

Note that for $g \in G$ the map $x \mapsto g \bullet x$ is a homeomorphism with inverse coming from g^{-1}.

Example 7.8.

(1) Trivial action : $g \cdot x = x$ for all $g \in G$ and $x \in X$.
(2) $G = \mathrm{GL}_n(\mathbb{R})$, $X = \mathbb{R}^n$, the action given by matrix multiplication $A \cdot x = Ax$. This induces $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ acting on S^{n-1}.
(3) Let G be a topological group and H a subgroup. Then H acts on G via $H \times G \to G$ given by $h \bullet g = hg$.
(4) If N is a normal subgroup of G, then G acts on N by $g \bullet n = gng^{-1}$.

(5) S^3 acts on $\mathbb{R}^4$ by quaternion multiplication. This induces an injection $T\colon S^3 \to \mathrm{SO}(4)$.
(6) S^3 acts on $\mathbb{R}^3$ by considering elements of $\mathbb{R}^3$ as imaginary quaternions $ai + bj + ck$ and using

$$x \bullet (ai + bj + ck) = x(ai + bj + ck)\bar{x}.$$

This induces a group homomorphism $C\colon S^3 \to \mathrm{SO}(3)$ which is onto and has kernel given by $\{1, -1\}$.

Definition 7.9. If G acts on X, we can define an equivalence relation $\sim$ on X by saying $x \sim y$ if there is $g \in G$ with $gx = y$. An equivalence class is called an *orbit*, and denoted by Gx. The corresponding quotient space is called the *orbit space*, denoted by X/G. If X/G is just a point, the action is called *transitive*.

Example 7.10.

(1) If G acts trivially, every orbit is a point and $X/G = X$.
(2) The action of $\mathrm{O}(n)$ on S^{n-1} is transitive. The action of $\mathrm{SO}(n)$ on S^{n-1} is transitive, provided that $n \geq 2$.
(3) $\mathbb{R}^n/\mathrm{GL}_n(\mathbb{R})$ consists of two points, the orbit of 0, and a non-zero orbit. The topology on the quotient is neither discrete nor indiscrete.

Theorem 7.11. *Let G be a connected topological group that acts on a topological space X such that X/G is connected. Then X is connected.*

Corollary 7.12. $\mathrm{SO}(n)$ *is connected for $n \geq 1$.*