Topology (Math 3281)

Solutions to Problem Class 1 27.10.14

1. Let $a \in A$ and $r > 0$. Let us write $B_A(a;r)$ for the ball of radius r in A using the metric d_A, and $B(a;r)$ for the ball of radius r around a in M. Then $B_A(a;r) = B(a;r) \cap A$. In particular, $B_A(a;r)$ is open in the subspace topology. As every open subset of A in the metric topology is a union of balls $B_A(a;r)$, we see that such U is also open in the subspace topology.

Now let $U \subset A$ be open in the subspace topology. Then there is an open set $V \subset M$ with $U = A \cap V$. As M has the metric topology, we can write

$$V = \bigcup_{x \in V} B(x; r_x)$$

where $r_x > 0$ is an appropriate radius. If $x \in V \cap A = U$, then $B_A(x;r_x) = B(x;r_x) \cap A \subset V \cap A$, so

$$U = \bigcup_{x \in U} B_A(x; r_x).$$

But this means that U is open in the metric topology on A coming from d_A.

2. Assume that X is finite. Then $\tau = \wp(X)$ and X is discrete, hence Hausdorff. If X is infinite, let $x, y \in X$ be two different points. Let $U \subset X$ be an open set containing x, and $V \subset X$ an open set containing y. Then $X - U$ and $X - V$ are both finite. Hence $X - (U \cap V) = (X - U) \cup (X - V)$ is also finite. But as X is infinite, $U \cap V$ has to have infinitely many elements, so it cannot be empty. Hence X is not Hausdorff in that case.

3. If X is discrete, there is a metric for the topology and X is Hausdorff. So now assume that X is Hausdorff. Let $x \in X$ be an element, and let $\{x_1, \ldots, x_k\} = X - \{x\}$. As X is Hausdorff, there exist open sets $U_1, \ldots, U_k$ and $V_1, \ldots, V_k$ with $x \in U_i$ and $x_i \in V_k$ for all $i = 1, \ldots, k$, with $U_i \cap V_i = \emptyset$ for all $i = 1, \ldots, k$. Now

$$x \in \bigcap_{i=1}^{k} U_i = U$$

is an open set containing x, and since $x_i \notin U_i \subset U$ for all $i = 1, \ldots, k$ we get that $U = \{x\}$. In particular, the set containing only x is open. This works for every $x \in X$, and therefore every subset of X is open, meaning that X is discrete.

4. Assume that there is a homeomorphism h between $\mathbb{R}$ with the standard topology τ and the discrete topology. Then h induces a bijection of $\wp(\mathbb{R})$ which sends the open sets of $\mathbb{R}$ in the standard topology to the open sets of $\mathbb{R}$ in the discrete topology. But the open sets in the discrete topology are $\wp(\mathbb{R})$, so this would mean that the open sets in the standard topology are also $\wp(\mathbb{R})$. However, that is not the case, as for example $\{0\}$ is not open. Hence there cannot be a homeomorphism.