Topology (Math 3281)

Solutions to Problem Class 2 10.11.14

1. (a) $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, so $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$ as $\overline{A \cup B}$ is the smallest such set.

Also $\overline{A \cup B}$ is a closed set containing A, so $\overline{A} \subset \overline{A \cup B}$. The same argument gives $\overline{B} \subset \overline{A \cup B}$. Therefore $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

Therefore $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(b) $\overline{A} \cap \overline{B}$ is a closed set containing $A \cap B$, so $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(c) $\overline{A}$ is closed, so it contains all of its limit points, so $\overline{\overline{A}} = \overline{A}$.

(d) Choose $A = (0, 1)$ and $B = (1, 2)$, then $A \cap B = \emptyset = \overline{A \cap B}$, but $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$, so $\overline{A} \cap \overline{B} = \{1\}$ strictly bigger than $\emptyset$.

2. (a) A° is an open set contained in $A \cup B$, so $A^\circ \subset (A \cup B)^\circ$. For the same reason $B^\circ \subset (A \cup B)^\circ$, so $A^\circ \cup B^\circ \subset (A \cup B)^\circ$.

(b) $(A \cap B)^\circ$ is an open subset contained in both A and B, so $(A \cap B)^\circ A^\circ \cap B^\circ$. Also, $A^\circ \cap B^\circ$ is an open set contained in $A \cap B$, so $A^\circ \cap B^\circ \subset (A \cap B)^\circ$.

(c) Since A° is open, it appears in the defining union of $(A^\circ)^\circ$ and so $A^\circ \subset (A^\circ)^\circ$. The other inclusion follows by definition.

(d) Take $A = (0, 1]$, $B = (1, 2)$. Then $A^\circ = (0, 1)$ and $B^\circ = (1, 2)$, so $1 \notin A^\circ \cup B^\circ$, while $A \cup B = (0, 2) = (A \cup B)^\circ$ containes 1.

3. Note that the complement of $\overline{A} \times \overline{B}$ can be written as $X - \overline{A} \times Y \cup X \times Y - \overline{B}$, so it is a closed set. As it contains $A \times B$, this gives $\overline{A \times B} \subset \overline{A} \times \overline{B}$. Now let a be a limit point of A, and $b \in \overline{B}$. Any neighborhood of (a, b) contains a neighborhood of the form $U \times V$ with U a neighborhood of a and V a neighborhood of b. Then U contains points of A different from a, and V contains points of B. Hence $U \times V$ contains points of $A \times B$ different from (a, b), and (a, b) is a limit point of $A \times B$. Similarly, any $(a, b) \in \overline{A} \times \overline{B}$ with b a limit point of B is a limit point of $A \times B$. Therefore $\overline{A} \times \overline{B} \subset \overline{A \times B}$.

4. (a) Let $d_P((x, x'), (y, y')) = 0$. Since d and d' are metrics, we get that $x = y$ and $x' = y'$, and also $d_P((x, x'), (x, x')) = 0$. Symmetry is also clear from the fact that d and d' are symmetric. The triangle inequality is also just using the triangle inequalities for both d and d'.

(b) We need to show that the open sets are the same. Since every open set U in the product topology is a union of sets of the form $B(x, r) \times B(x', r')$, where $(x, x') \in U$ and $r, r' > 0$ depend on (x, x'), and every open set V in the metric topology coming from d_P is a union of sets of the form $B((x,x'),R)$, where $(x,x') \in V$ and $R > 0$ appropriately, we need to find $R > 0$ with

$$B((x,x'),R) \quad \subset \quad B(x,r) \times B(x',r') \tag{1}$$

for given $r, r' > 0$, and we need to find $r, r' > 0$ with

$$B(x,r) \times B(x',r') \quad \subset \quad B((x,x'),R) \tag{2}$$

for given $R > 0$.

So let $r, r' > 0$ be given. Then let $R = \min\{r, r'\} > 0$. If $d_P((x,x'),(y,y') < R$, then both $d(x,y) < r$ and $d'(x',y') < r'$, so $(y,y') \in B(x,r) \times B(x',r')$. This shows (1).

Given $R > 0$, let $r = r' = R/2$. Then if $d(x,y) < r$ and $d'(x',y') < r' = r$, then there sum is less than R, so $(y,y') \in B((x,x'),R)$, and (2) follows.