Topology (Math 3281)

Solutions to Problem Class 3 24.11.14

1. (a) Note that $A_{ij}(\lambda)$ differs from the identity matrix in only one off-diagonal entry. Define $\gamma : [0,1] \to GL_n(\mathbb{R})$ by $\gamma(t) = A_{ij}(t\lambda)$. Then $\gamma(0) = I$, $\gamma(1) = A_{ij}(\lambda)$ and the determinant is constant to 1 during the path.

(b) Define $\gamma : [0,1] \to GL_n(\mathbb{R})$ by $\gamma(t) = M_i(t\mu + (1-t))$. The determinant of $\gamma(t)$ is $t\mu + (1-t) = 1 + t(\mu - 1)$. As $\mu > 0$ and $t \in [0,1]$, this determinant is always bigger than 0, so each element in the path is an invertible matrix.

(c) Write $A = E_1 \cdots E_k$ with E_m either of the form $A_{ij}(\lambda)$ or $M_i(\mu)$. Now let γ_m be a path from E_m to I, unless $E_m = M_i(\mu)$ with $\mu < 0$, in which case we can choose a path to $M_i(-1)$ with a similar argument as in (b). Then matrix-multiplication of these paths gives a path in $GL_n(\mathbb{R})$ from A to a diagonal matrix J whose entries are either $+1$ or -1. As the determinant of J has to be the same sign as $\det A$, we have to have an even number of -1 in J. Now there is a path in $GL_n(\mathbb{R})$ which cancels two -1 in J. This can be seen by looking at the special case of $J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

As a path, use

$$\gamma(t) = \begin{pmatrix} \cos \pi t & \sin \pi t \\ -\sin \pi t & \cos \pi t \end{pmatrix}$$

After finitely many of these paths, we get a path from A to I.

(d) The argument over $\mathbb{C}$ is essentially identical, with the exception that for all $M_i(\mu)$ with $\mu \in \mathbb{C} - \{0\}$ we can find a path to the identity matrix.

2. (a) Let $x, y \in \bigcup A_i$. Then there is $i, j \in I$ with $x \in A_i$ and $y \in A_j$. Since $A_i \cap A_j$ is path connected, there is a path from x to y, which is also a path in $\bigcup A_i$. It would be enough to assume that $A_i \cap A_j$ is non-empty, as we then can find a path from x to $z \in A_i \cap A_j$ in A_i, and a path from y to z in A_j, which can be combined to a path in $\bigcup A_i$ from x to y.

(b) We define C to be a path component if it is a maximal path component subset of X. If C_1 and C_2 are path components with $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is path connected by (improved version of) (a), so by maximality $C_1 = C_2$. If $x \in X$, let

$$C_x = \bigcup_C C$$

where the union is taken over all path connected subsets containing x. Again, by the improved version of (a) we get that this is path connected and it is maximal by construction.