Topology (Math 3281)

Solutions to Problem Class 4 08.12.14

1. Define $m : \mathbb{Z}_2 \times S^1 \times S^1 \to S^1 \times S^1$ by $m(\bar{1}, z, w) = (\bar{z}, w)$ and $m(\bar{0}, z, w) = (z, w)$. This is clearly an action. Denote the quotient space by X. Define $f : X \to S^1 \to [0, 1]$ by $f([z, w]) = (w, \Re z)$, where $\Re z$ denotes the real part of $z \in \mathbb{C}$. Since $\Re z = \Re \bar{z}$, this is a well defined function, which is easily seen to be bijective. As X is compact and $S^1 \times [0, 1]$ a Hausdorff space, this is a homeomorphism.

Remark: To see why this action works, note that the second factor only goes for the ride. We therefore need an action such that the quotient space of S^1 is the interval. If one pictures the circle in the complex plane, you can see that flipping along an axis should do the right thing. The above action formalizes this idea.

2. (a) We need $\pi^{-1}(\pi(U)) = \bigcup_{g\in G} gU$. Let $x \in \pi^{-1}(\pi(U))$. Then $\pi(x) \in \pi(U)$, hence there is a $u \in U$ with $Gx = Gu$. Therefore there is a $g \in G$ with $gx = u$, and $x \in g^{-1}U$. For the other inclusion, let $x \in \bigcup_{g\in G} gU$. Then there is a $g \in G$ with $x \in gU$, which means there is a $u \in U$ with $x = gu$. Then $\pi(x) = \pi(gu) = \pi(u) \in \pi(U)$, so $x \in \pi^{-1}(\pi(U))$.

Now if $U \subset X$ is open, in order for $\pi(U)$ to be open we need $\pi^{-1}(\pi(U))$ to be open by the definition of the quotient topology. But as each gU is also open (recall left multiplication is a homeomorphism of X), this is a union of open sets by the previous result.

(b) Let $GL_n(\mathbb{R})$ act on $\mathbb{R}^n$ as in the lectures. The quotient space is a non-Hausdorff space containing the orbit of 0 and the orbit of everything else. If $x \in \mathbb{R}^n$ is non-zero, $\{x\} \subset \mathbb{R}^n$ is closed, but $\pi(\{x\})$ is not closed, as the complement, consisting just of the orbit of 0, is not open.

3. Let us first show that $U(n)$ is a topological group. If $A, B \in U(n)$, then $(AB)^* = B^*A^*$ and $AB(AB)^* = ABB * A* = I$, so $AB \in U(n)$. As $A \in GL_n(\mathbb{C})$, the inverse A^{-1} exists and is A^*. In particular, $A^*A = I$, so the inverse of $A \in U(n)$ is in $U(n)$. Multiplication and taking inverses are continuous, associativity holds as it holds for matrix multiplication, therefore

$U(n)$ is a topological group. To see that $U(n)$ is compact, note that

$$\begin{aligned} 1 &= \sum_{j=1}^{n} a_{ij} a_{ji}^* \\ &= \sum_{j=1}^{n} a_{ij} \bar{a}_{ij} \\ &= \sum_{j=1}^{n} |a_{ij}|^2 \end{aligned}$$

which implies that each $|a_{ij}|^2 \leq 1$. Hence $U(n)$ is a bounded subset of $C^{n^2} \cong \mathbb{R}^{4n^2}$. Furthermore, $U(n) = g^{-1}(\{I\})$, where $g : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$ is given by $g(A) = AA^*$. Hence $U(n)$ is also a closed subset of $\mathbb{R}^{4n^2}$, which means that $U(n)$ is compact by Heine-Borel.