1. Quaternions

Definition 1.1. The skew-field of *Quaternions* is defined as $\mathbb{H} = \mathbb{C} \times \mathbb{C}$, where addition is the addition of the vectorspace $\mathbb{C} \times \mathbb{C}$, and multiplication is defined as

$$(z_1, z_2) \cdot (w_1, w_2) = (z_1 w_1 - \bar{w}_2 z_2, w_2 z_1 + z_2 \bar{w}_1). \tag{1}$$

Remark 1.2. One can think of $(z_1, z_2) \equiv z_1 + z_2 j$, where j is a symbol as $i \in \mathbb{C}$. Notice that $j^2 = -1$ and $ij = -ji$. In particular, this multiplication is not commutative.
Writing $k = ij$, one can think of $\mathbb{H} = \mathbb{R}^4$, and elements can be written as $a + bi + cj + dk$ with $a, b, c, d \in \mathbb{R}$.

We stated that the quaternions are a skew-field, which is a ring such that every non-zero element has a multiplicative inverse. This actually requires some work, however we only have to show ring axioms involving the multiplication, as additively $\mathbb{H}$ is a $\mathbb{C}$-vectorspace.

Lemma 1.3. *Let $a \in \mathbb{R}$, $z \in \mathbb{C}$, $x, x_1, x_2, x_3 \in \mathbb{H}$. Then the following hold.*

(1) $a \cdot x = x \cdot a$.
(2) $z \cdot j = j \cdot \bar{z}$.
(3) $x_1 \cdot (x_2 \cdot x_3) = (x_1 \cdot x_2) \cdot x_3$.
(4) $x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$.
(5) $(x_1 + x_2) \cdot x_3 = x_1 \cdot x_3 + x_2 \cdot x_3$.

Proof. Write $a = (a, 0)$ and $x = (z_1, z_2)$. Then

$$\begin{aligned} (a,0) \cdot (z_1, z_2) &= (az_1, z_2 a) \\ &= (z_1 a, a z_2) = (z_1, z_2) \cdot (a, 0). \end{aligned}$$

For (2), note that

$$\begin{aligned} (z,0) \cdot (0,1) &= (0, z) \\ (0,1) \cdot (z,0) &= (0, \bar{z}). \end{aligned}$$

We now write $x_m = z_m + w_m j$ for $m = 1, 2, 3$ with $z_m, w_m \in \mathbb{C}$. Then

$$(z_1 + w_1 j)(z_2 + w_2 j) = z_1 z_2 - w_1 \bar{w}_2 + (z_1 w_2 + w_1 \bar{z}_2) j,$$

but notice that this used that complex multiplication is commutative. In particular, quaternion multiplication agrees with the naive multiplication obtained by distributing complex numbers with rule (2). Checking (3), (4) and (5) is then just a matter of putting the obvious symbols together. We omit the details. □

To get inverses, we define a conjugation for $\mathbb{H}$.

Definition 1.4. The *conjugate* of a quaternion (z_1, z_2) is defined as

$$\overline{(z_1, z_2)} = (\bar{z}_1, -z_2).$$

The *norm* of a quaternion (z_1, z_2) is defined as

$$|(z_1, z_2)| \;=\; \sqrt{|z_1|^2 + |z_2|^2}.$$

Remark 1.5. If we write a quaternion as $a + bi + cj + dk$, the conjugate is $a - bi - cj - dk$, and the norm is $\sqrt{a^2 + b^2 + c^2 + d^2}$. In particular the norm is the euclidean distance from the origin in $\mathbb{R}^4$.

Lemma 1.6. *For $x \in \mathbb{H}$ we have*

$$x \cdot \bar{x} \;=\; |x|^2 \;=\; \bar{x} \cdot x.$$

Proof. Let $x = (z_1, z_2)$, then

$$\begin{aligned} x \cdot \bar{x} &= (z_1, z_2) \cdot (\bar{z}_1, -z_2) \\ &= (z_1 \bar{z}_1 + \bar{z}_2 z_2, -z_2 z_1 + z_2 z_1) \\ &= (|z_1|^2 + |z_2|^2, 0) \\ &= |x|^2. \end{aligned}$$

To see that $|x|^2 = \bar{x} \cdot x$, note that $\bar{\bar{x}} = x$ and $|x| = |\bar{x}|$. □

Proposition 1.7. *The set $\mathbb{H} - \{0\}$ with multiplication is a topological group, the inverse of $x \neq 0$ is given as*

$$x^{-1} \;=\; \frac{1}{|x|^2}\,\bar{x}.$$

Proof. Multiplication and the formula for the inverse are clearly continuous. Associativity is satisfied by Lemma 1.3. To see that the formula really gives the inverse, note that

$$\begin{aligned} x \cdot \frac{1}{|x|^2}\,\bar{x} &= \frac{1}{|x|^2} x \cdot \bar{x} \\ &= \frac{|x|^2}{|x|^2} \\ &= 1 \\ &= \frac{1}{|x|^2}\,\bar{x} \cdot x \end{aligned}$$

where we used Lemma 1.3 and Proposition 1.7. □

Lemma 1.8. *If $x, y \in \mathbb{H}$, then*

$$|x \cdot y| \;=\; |x| \cdot |y|.$$

Proof. Let $x = (z_1, z_2)$ and $y = (w_1, w_2)$. Then

$$\begin{aligned} |x \cdot y| &= \sqrt{(z_1 w_1 - \bar{w}_2 z_2)(\bar{w}_1 \bar{z}_1 - \bar{z}_2 w_2) + (w_2 z_1 + z_2 \bar{w}_1)(\bar{z}_1 \bar{w}_2 + w_1 \bar{z}_2)} \\ &= \sqrt{z_1 w_1 \bar{w}_1 \bar{z}_1 + \bar{w}_2 z_2 \bar{z}_2 w_2 + w_2 z_1 \bar{z}_1 \bar{w}_2 + z_2 \bar{w}_1 w_1 \bar{z}_2} \\ &= \sqrt{|z_1|^2 |w_1|^2 + |z_2|^2 |w_2|^2 + |z_1|^2 |w_2|^2 + |z_2|^2 |w_1|^2} \\ &= \sqrt{(|z_1|^2 + |z_2|^2)(|w_1|^2 + |w_2|^2)} \\ &= |x| \cdot |y| \end{aligned}$$

as was to be shown. $\square$

Corollary 1.9. *The set S^3 with quaternion multiplication is a subgroup of $\mathbb{H} - \{0\}$, furthermore, the inverse of $x \in S^3$ is $\bar{x}$.*

Proof. By Lemma 1.8, we get $x \times y \in S^3$ for $x, y \in S^3$, so the set is closed under multiplication. Clearly, $1 \in S^3$ and since $x \cdot \bar{x} = |x|^2 = 1$ for $x \in S^3$, the inverse of $x \in S^3$ is just given as $\bar{x}$. $\square$

Lemma 1.10. *Let $x, y \in \mathbb{H}$, then*

$$\overline{x \cdot y} = \bar{y} \cdot \bar{x}.$$

Proof. Let $x = (z_1, z_2)$ and $y = (w_1, w_2)$. Then

$$\begin{aligned} \overline{x \cdot y} &= (\overline{z_1 w_1 - \bar{w}_2 z_2}, -w_2 z_1 - z_2 \bar{w}_1) \\ &= (\bar{w}_1 \bar{z}_1 - \bar{z}_2 w_2, -w_2 z_1 - z_2 \bar{w}_1) \end{aligned}$$

and

$$\bar{y} \cdot \bar{x} = (\bar{w}_1 \bar{z}_1 - \bar{z}_2 w_2, -z_2 \bar{w}_1 - w_2 z_1).$$

Since addition of complex numbers is commutative, the result follows. $\square$

Remark 1.11. In the proof of Lemma 1.8 we used that

$$0 = -z_1 w_1 \bar{z}_2 w_2 - \bar{w}_2 z_2 \bar{w}_1 \bar{z}_1 + w_2 z_1 w_1 \bar{z}_2 + z_2 \bar{w}_1 \bar{z}_1 \bar{w}_2$$

which is obvious, as all variables are complex numbers so we can use commutativity. If we would think of these variables as quaternions, we could not use commutativity, however, we could write

$$\begin{aligned} a &= z_1 w_1 \bar{z}_2 w_2 \\ b &= w_2 a w_2^{-1} \end{aligned}$$

and then

$$-z_1 w_1 \bar{z}_2 w_2 - \bar{w}_2 z_2 \bar{w}_1 \bar{z}_1 + w_2 z_1 w_1 \bar{z}_2 + z_2 \bar{w}_1 \bar{z}_1 \bar{w}_2 = -a - \bar{a} + b + \bar{b}.$$

Then

$$\begin{aligned} -(a + \bar{a}) + b + \bar{b} &= -(a + \bar{a}) + w_2 (a + \bar{a}) w_2^{-1} \\ &= -(a + \bar{a}) + w_2 w_2^{-1} (a + \bar{a}) \\ &= 0, \end{aligned}$$

since $a + \bar{a} \in \mathbb{R}$ for every $a \in \mathbb{H}$, so it commutes with $w_2 \in \mathbb{H}$. So commutativity is not needed in the proof of Lemma 1.8.

The point is that one can define *Octonions* as pairs of quaternions, and a multiplication of octonions is defined by the formula (1). Most of the results for quaternion multiplication carry over, for example the analogue of Lemma 1.8 by the argument above. However, the multiplication is not associative. In particular, S^7 is invariant under this multiplication, but it is not a topological group.

Lemma 1.12. *Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of $\mathbb{R}^4$. Then for $x, y \in \mathbb{H}$ we have*

$$\langle x, y \rangle \quad = \quad \frac{1}{2}(x \cdot \bar{y} + y \cdot \bar{x}).$$

Proof. By Lemma 1.6 we have

$$\langle x+y, x+y \rangle = (x+y)(\overline{x+y}) = x\bar{x} + x\bar{y} + y\bar{x} + y\bar{y}.$$

Using bilinearity of the inner product, we also get

$$\langle x+y, x+y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle.$$

With Lemma 1.6 again, we get the result. $\square$

Example 1.13. The topological group S^3 acts on $\mathbb{H}$ as follows: Let $x \in S^3$ and $y \in \mathbb{H}$, then define

$$x \bullet y \quad = \quad x \cdot y \cdot \bar{x},$$

which is easily seen to be a group action. Note that the action is given by conjugation of y by $x \in S^3$. For $x \in S^3$ we will now write $C_x : \mathbb{H} \to \mathbb{H}$ given by $C_x(y) = x \cdot y \cdot \bar{x}$. It is clear that C_x is a linear map, and in fact the assignment

$$C : S^3 \to \mathrm{GL}_4(\mathbb{R})$$

given by $C(x) = C_x$ is a group homomorphism. We claim that the image is in fact contained in $SO(4)$. To see this, we need to make sure that each C_x preserves the inner product. Let $a, b \in \mathbb{H}$ and $x \in S^3$. Then

$$\begin{aligned} \langle C_x(a), C_x(b) \rangle &= \langle xa\bar{x}, xb\bar{x} \rangle \\ &= \frac{1}{2}(xa\bar{x}x\bar{b}\bar{x} + xb\bar{x}x\bar{a}\bar{x}) \\ &= \frac{1}{2}(x(a\bar{b} + b\bar{a})\bar{x}) \\ &= x\langle a, b \rangle \bar{x} \\ &= \langle a, b \rangle, \end{aligned}$$

because elements of $\mathbb{R}$ (such as $\langle a, b \rangle$) commute with $x \in \mathbb{H}$. This proves that each $C_x \in O(4)$. But clearly C_1 is the identity, and since S^3 is connected, we have a homomorphism

$$C : S^3 \to SO(4).$$

Notice however that if $a \in \mathbb{R}$, then $C_x(a) = a$, so C_x keeps the 1-dimensional subspace $\mathbb{R}$ of $\mathbb{H}$ invariant, and therefore also its orthogonal complement, which is isomorphic to $\mathbb{R}^3$. So if we restrict the linear map C_x to this copy of $\mathbb{R}^3$, we actually get a homomorphism

$$C : S^3 \to SO(3).$$

We now want to show that this homomorphism is indeed surjective.

Definition 1.14. Let $a \in \mathbb{H}$. We then define the *real part* of a as

$$\Re a = \frac{1}{2}(a + \bar{a})$$

and the *imaginary part* of a as

$$\Im a = \frac{1}{2}(a - \bar{a}).$$

An element $a \in \mathbb{H}$ is called *imaginary*, if $\Re a = 0$.

Remark 1.15. Notice that indeed $\Re a \in \mathbb{R}$, and $\Re(\Im a) = 0$. The imaginary numbers of S^3 form a sphere S^2, and for imaginary $x \in S^3$ we get that $x^{-1} = -x$. In particular, the polynomial $p(x) = x^2 + 1$ has a whole S^2 as roots, when viewed over $\mathbb{H}$.

Now we can think of S^3 acting on the imaginary quaternions, which form a 3-dimensional vectorspace, by conjugation.

Theorem 1.16. *The homomorphism $C : S^3 \to SO(3)$ is surjective.*

The main work in this theorem is to show that $SO(3)$ is not "too big". Before we show this, let us take a look at $SO(2)$ and $O(2)$.

Lemma 1.17. *The homomorphism $H : S^1 \to SO(2)$ given by*

$$H(e^{i\theta}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is an isomorphism.

Proof. Since $AA^t = I$ for all $A \in SO(2)$, we get that any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has to satisfy

$$\begin{aligned} a^2 + b^2 &= 1 \\ c^2 + d^2 &= 1 \\ ac + bd &= 0 \end{aligned}$$

and since $\det A = 1$, we also get

$$ad + bc = 1.$$

It is now easy to see that each $A \in SO(2)$ has to look as in the statement of the lemma, that is, that there is a $\theta \in [0, 2\pi]$ with $H(e^{i\theta}) = A$. This map is also clearly injective, so that H is a homeomorphism. To see that H is an isomorphism, one has to use the standard properties of trigonometric functions which express sine and cosine of $\theta + \theta'$. $\square$

Remark 1.18. So every element of $SO(2)$ is a rotation by some angle θ. Notice in particular, that for $\theta \neq 0, \pi$ this rotation has no real eigenvalues.

All elements of $O(2)$ with determinant -1 can now be described as rotations followed by a reflection in the x-axis. Using linear algebra, it is now easy to see that each element of $O(2)$ with determinant -1 has an eigenvalue -1 and an eigenvalue $+1$.

Lemma 1.19. *Let $A \in SO(3)$. Then A has at least one eigenvalue equal to 1. Furthermore, any $A \in SO(3)$ different from I can be described as fixing a one-dimensional eigenspace, and rotating the orthogonal plane to this eigenspace.*

Proof. The characteristic polynomial $\chi_A(x)$ has degree 3, so it has a real zero by the Intermediate Value Theorem. Denote this root by $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^3$ be an eigenvector. Then

$$\begin{aligned} \langle x, x \rangle &= \langle Ax, Ax \rangle \\ &= \langle \lambda x, \lambda x \rangle \\ &= \lambda^2 \langle x, x \rangle, \end{aligned}$$

so $\lambda \in \{\pm 1\}$. If $\lambda = -1$, then A restricted to the 2-dimensional orthogonal complement of x can be identified with an element $B \in O(2)$. Note that $\det B = -1$, for otherwise $\det A = -1$. Hence B has an eigenvalue $+1$ by Remark 1.18, and so does A.

Therefore A fixes a one-dimensional subspace $\langle y \rangle$ of $\mathbb{R}^3$. The restriction to the orthogonal complement gives an element of $SO(2)$, which is a rotation by Lemma 1.17. This proves the result. □

Proof of Theorem 1.16. Let $A \in SO(3)$, and let $u \in S^2$ be fixed by A, as guaranteed by Lemma 1.19. Also, let $2\theta \in [0, 2\pi]$ be the angle of rotation of the plane orthogonal to u. Let $x = \cos\theta + \sin\theta \cdot u \in S^3$.

We claim that C_x fixes u and rotates the plane orthogonal to u by an angle of 2θ.

To see this first note that $\bar{x} = \cos\theta - \sin\theta \cdot u$, so

$$\begin{aligned} x \cdot u \cdot \bar{x} &= (\cos\theta + \sin\theta \cdot u) \cdot u \cdot (\cos\theta - \sin\theta \cdot u) \\ &= \cos^2\theta \cdot u - \cos\theta \sin\theta \cdot u^2 + \sin\theta \cdot u^2 \cos\theta - \sin^2\theta \cdot u^3 \\ &= (\cos^2\theta + \sin^2\theta) \cdot u \\ &= u \end{aligned}$$

and x fixes u. Notice that $u^3 = -u$, as u is imaginary.

Now choose an imaginary $v \in S^2$ which is orthogonal to u. This means $u\bar{v} = -v\bar{u}$ by Lemma 1.12. We claim that uv is also imaginary. To see this note that

$$\begin{aligned} uv + \overline{uv} &= uv + \bar{v}\bar{u} \\ &= uv - v\bar{u} \\ &= uv + u\bar{v} \\ &= uv - uv \\ &= 0, \end{aligned}$$

where we used that $\bar{v} = -v$ and $u\bar{v} = -v\bar{u}$ by orthogonality.

Also, uv is orthogonal to both u and v:

$$\begin{aligned}\langle u, uv\rangle &= \frac{1}{2}(u\bar{v}\bar{u} + uv\bar{u})\\ &= \frac{1}{2}(-uv\bar{u} + uv\bar{u})\\ &= 0.\\ \langle v, uv\rangle &= \frac{1}{2}(v\bar{v}\bar{u} + uv\bar{v})\\ &= \frac{1}{2}(\bar{u} + u)\\ &= 0.\end{aligned}$$

as u and v are imaginary. Hence v and uv span the plane orthogonal to u. We now need to check that C_x rotates v and uv by an angle of 2θ. Now

$$\begin{aligned}C_x(v) &= (\cos\theta + \sin\theta\cdot u)v(\cos\theta - \sin\theta\cdot u)\\ &= \cos^2\theta\cdot v - \cos\theta\sin\theta\cdot vu + \sin\theta\cos\theta\cdot uv - \sin^2\theta\cdot uvu\\ &= (\cos^2\theta - \sin^2\theta)\cdot v + 2\cos\theta\sin\theta\cdot uv\\ &= \cos(2\theta)\cdot v + \sin(2\theta)\cdot uv,\end{aligned}$$

where we used that $vu = -uv$, which follows from orthogonality and the fact that u and v are imaginary. Similarly

$$\begin{aligned}C_x(uv) &= (\cos\theta + \sin\theta\cdot u)uv(\cos\theta - \sin\theta\cdot u)\\ &= \cos^2\theta\cdot uv - \cos\theta\sin\theta\cdot uvu + \sin\theta\cos\theta\cdot uuv - \sin^2\theta\cdot uuvu\\ &= (\cos^2\theta - \sin^2\theta)\cdot uv - 2\cos\theta\sin\theta\cdot v\\ &= -\sin(2\theta)\cdot v + \cos(2\theta)\cdot uv.\end{aligned}$$

Therefore $C_x = A$, and the homomorphism is surjective. $\square$

Corollary 1.20. *The topological group $\mathbb{R}\mathbf{P}^3 = S^3/\{\pm 1\}$ is isomorphic to $SO(3)$.*

Proof. We need to show that the normal subgroup $\{\pm 1\}$ of S^3 is the kernel of C. Clearly, it is contained in the kernel. Now let $x \in S^3$ satisfy $C_x = I$. Then x commutes with every imaginary $a \in \mathbb{H}$, and since it commutes with every real number, we get that $xy = yx$ for all $y \in \mathbb{H}$. Write $x = z + wj$. Then

$$\begin{aligned}x\cdot j &= -w + zj\\ j\cdot x &= -\bar{w} + \bar{z}j.\end{aligned}$$

Which implies that $z, w \in \mathbb{R}$. Also

$$\begin{aligned}x\cdot i &= zi - wij\\ i\cdot x &= iz + wij,\end{aligned}$$

which implies that $w = -w$, that is, $w = 0$. Hence $x \in \mathbb{R}$, and since $\mathbb{R}\cap S^3 = \{\pm 1\}$, this is the kernel of C. $\square$