2 Branching processes

2.1 Classification and extinction

Informally, a branching process [7] is described as follows, where $\{p_k\}_{k\ge 0}$ is a fixed probability mass function (pmf).

- A population starts with a single ancestor who forms generation number 0.
- This initial individual splits into k offspring with probability p_k for each $k \ge 0$; the resulting offspring constitute the first generation.
- Each of the offspring in the first generation splits independently into a random number of offspring, again according to the pmf $\{p_k\}_{k\ge 0}$, and the resulting offspring constitute the second generation.
- This process continues until extinction, which occurs when all the members of a generation fail to produce offspring.

This model has a number of applications in biology (eg., it can be thought as a model of population growth), physics (chain reaction in nuclear fission), queueing theory, etc. Originally it arose from a study of the likelihood of survival of family names ("how fertile must a family be to ensure that in future generations the family name will not die out?").

Formally, let $\{Z_{n,k}\}$, $n \ge 1$, $k \ge 1$, be a family of i.i.d. random variables in $\mathbb{Z}^+$, each having a common probability mass function $\{p_k\}_{k\ge 0}$. Then the branching process $(Z_n)_{n\ge 0}$ (generated by $\{p_k\}_{k\ge 0}$) is defined by setting $Z_0 = 1$, and, for $n \ge 1$,

$$Z_n \stackrel{\mathsf{def}}{=} Z_{n,1} + Z_{n,2} + \cdots + Z_{n,Z_{n-1}}\,, \tag{2.1}$$

where the empty sum is interpreted as zero. Notice that Z_n is a Markov chain in $\mathbb{Z}^+$. We shall use $\mathsf{P}(\,\cdot\,) \equiv \mathsf{P}_1(\,\cdot\,)$ and $\mathsf{E}(\,\cdot\,) \equiv \mathsf{E}_1(\,\cdot\,)$ to denote the corresponding probability measure and the expectation operator.[8] If $\varphi_n(s) \equiv \mathsf{E}(s^{Z_n})$ is the generating function of Z_n, a straightforward induction based on (2.1) and (1.5) implies that

$$\varphi_0(s) \equiv s\,, \qquad \varphi(s) \equiv \varphi_1(s) \equiv \mathsf{E}s^{Z_1}\,,$$
$$\varphi_k(s) = \varphi_{k-1}\big(\varphi(s)\big) \equiv \varphi\big(\varphi_{k-1}(s)\big) = \underbrace{\varphi(\ldots\varphi(s)\ldots)\ldots)}_{k \text{ times}} \quad k > 1. \tag{2.2}$$

Usually explicit calculations are hard, but at least in principle, equations (2.2) determine the distribution of Z_n for any $n \ge 0$.

Example 2.1. *Let $\varphi_1(s) \equiv \varphi(s) = q + ps$ for some $0 < p = 1 - q < 1$. Then*

$$\varphi_n(s) \equiv q(1 + p + \cdots + p^{n-1}) + p^n s = 1 + p^n(s-1)\,.$$

Notice that here we have $\varphi_n(s) \to 1$ as $n \to \infty$, for all $s \in [0,1]$. In other words, the distribution of Z_n converges to that of $Z_\infty \equiv 0$, recall Theorem 1.34.

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[7] sometimes called a Galton-Watson-Bienaymé process

[8] If $Z_0 = k$, we shall explicitly write $\mathsf{P}_k(\,\cdot\,)$ and $\mathsf{E}_k(\,\cdot\,)$.

The following result is a straightforward corollary of (1.5).

Lemma 2.2. *In a branching process $(Z_n)_{n\geq 0}$ with $Z_0 = 1$, let the offspring distribution have mean m. Then $\mathsf{E}(Z_n) = m^n$ for all $n \geq 1$.*

Proof. Recall that a random variable X with values in $\mathbb{Z}_{\geq 0}$ has finite mean equal to $G'_X(1_-) := \lim_{s\uparrow 1} G'_X(s)$, if and only if this limit exists and is finite. Since Z_1 is assumed to have finite mean m, this implies that $\varphi'(1_-) := \lim_{s\uparrow 1} \varphi'(s) = m$ (where φ is the generating function of Z_1). We also know by (1.5) that the generating function of Z_n is given by φ_n which is just the composition of φ with itself n times. By the chain rule, and since $\varphi_k(1) = 1$ for all k, we see that

$$\lim_{s\uparrow 1} \varphi'_n(1_-) = \varphi'(1_-)^n = m^n,$$

implying the result. This can alternatively be shown by induction, using a conditioning argument. □

Exercise 2.3. *In a branching process $(Z_n)_{n\geq 0}$ with $Z_0 = 1$, let the offspring distribution have mean m, variance σ^2, and generating function φ. Write φ_n for the generating function of the nth generation size Z_n, $\varphi_n(s) \equiv \mathsf{E}(s^{Z_n})$.*

(a) Using (2.2) or otherwise, show that $\mathsf{Var}(Z_n) = \sigma^2 m^{n-1}(m^n - 1)/(m-1)$ if $m \neq 1$ and $\mathsf{Var}(Z_n) = \sigma^2 n$ if $m = 1$.

(b) Deduce that $\mathsf{E}\big((Z_n/m^n)^2\big)$ is uniformly bounded for $m \neq 1$.

This result suggests that if $m \equiv \mathsf{E}(Z_1) \neq 1$, the branching process might explode (for $m > 1$) or die out (for $m < 1$). One therefore classifies branching process as either critical (if $m = 1$), subcritical ($m < 1$), or supercritical ($m > 1$).

Example 2.4. *It is straightforward to describe the case $m < 1$. Indeed, the Markov inequality (1.2) implies that*

$$\mathsf{P}(Z_n > 0) = \mathsf{P}(Z_n \geq 1) \leq \mathsf{E}(Z_n) = m^n\,,$$

so that $\mathsf{P}(Z_n > 0) \to 0$ as $n \to \infty$ (ie., $Z_n \to 0$ in probability). We also notice that the average total population *in this case is finite, $\mathsf{E}\big(\sum_{n\geq 0} Z_n\big) = \sum_{n\geq 0} m^n = (1-m)^{-1} < \infty$.*

Definition 2.5. *The extinction event $\mathcal{E}$ is the event $\mathcal{E} = \cup_{n=1}^{\infty}\big\{Z_n = 0\big\}$. Since $\big\{Z_n = 0\big\} \subset \big\{Z_{n+1} = 0\big\}$ for all $n \geq 0$, the extinction probability ρ is defined as*

$$\rho = \mathsf{P}(\mathcal{E}) = \lim_{n\to\infty} \mathsf{P}\big(Z_n = 0\big)\,,$$

where $\mathsf{P}\big(Z_n = 0\big) \equiv \varphi_n(0)$ is the extinction probability before $(n+1)$*st generation.*

The following result helps to derive the extinction probability ρ without needing to compute the iterates $\varphi_n(\,\cdot\,)$ precisely. To avoid trivialities we shall assume that $p_0 = \mathsf{P}(Z = 0)$ satisfies[9] $0 < p_0 < 1$; notice that under this assumption $\varphi(s)$ is a strictly increasing function of $s \in [0, 1]$.

Theorem 2.6. *If $0 < p_0 < 1$, then the extinction probability ρ is given by the smallest positive solution to the equation*

$$s = \varphi(s)\,. \tag{2.3}$$

In particular, if $m = \mathsf{E}Z_1 \le 1$, then $\rho = 1$; otherwise, we have $0 < \rho < 1$.

In words, if the branching process is subcritical or critical then it eventually becomes extinct with probability one. However, if it is supercritical, the process has a positive probability to survive for all time.

Remark 2.7. *There is a clear probablistic intuition behind the relation $\rho = \varphi(\rho)$. Indeed, if $\rho = \mathsf{P}_1(\mathcal{E})$ is the extinction probability starting from a single individual, $Z_0 = 1$, then by independence we get $\mathsf{P}_k(\mathcal{E}) \equiv \mathsf{P}(\mathcal{E} \mid Z_0 = k) = \rho^k$, and thus the first step decomposition for the Markov chain Z_n gives*

$$\begin{aligned}\rho = \mathsf{P}(\mathcal{E}) &= \sum_{k\ge 0} \mathsf{P}(\mathcal{E}, Z_1 = k) = \sum_{k\ge 0} \mathsf{P}(\mathcal{E} \mid Z_1 = k)\mathsf{P}(Z_1 = k) \\ &= \sum_{k\ge 0} \rho^k\, \mathsf{P}(Z_1 = k) \equiv \mathsf{E}\big(\rho^{Z_1}\big) \equiv \varphi(\rho)\,,\end{aligned}$$

in agreement with (2.3).

Proof. Let us now give the proof of Theorem 2.6. You may find it helpful to draw a picture!

Denote $\rho_n = \mathsf{P}\big(Z_n = 0\big) \equiv \varphi_n(0)$. By continuity and strict monotonicity of $\varphi(\,\cdot\,)$ we have (recall (2.2))

$$0 < \rho_1 = \varphi(0) < \rho_2 = \varphi(\rho_1) < \cdots < 1\,,$$

so that the extinction probability $\rho \in (0, 1]$ is the increasing limit of ρ_n as $n \to \infty$, and satisfies

$$\rho = \lim_n \varphi_n(0) = \lim_n \varphi(\varphi_{n-1}(0)) = \varphi(\lim_n \varphi_{n-1}(0)) = \varphi(\rho).$$

On the other hand, if $\bar\rho$ is any other fixed point of $\varphi(\,\cdot\,)$ in $[0, 1]$, ie., $\bar\rho = \varphi(\bar\rho)$, then $\bar\rho = \varphi_n(\bar\rho) \ge \varphi_n(0)$ for all n, meaning that $\bar\rho \ge \lim_{n\to\infty} \varphi_n(0) = \rho$. So, ρ is indeed the smallest positive solution to (2.3).

Next, we turn to the extinction criterion in terms of m. For this, observe that $\varphi(\,\cdot\,)$ is convex on $[0, 1]$, since $\varphi''(s) = \mathsf{E}(Z_1(Z_1 - 1)s^{Z_1-2}) \ge 0$ for $s \in [0, 1]$ (actually unless $p_1 = 1 - p_0$, i.e., if there is some possibility of having more than

[9] otherwise the model is degenerate: if $p_0 = 0$, then $Z_n \ge 1$ for all $n \ge 0$ so that $\rho = 0$; if $p_0 = 1$, then $\mathsf{P}(Z_1 = 0) = \rho = 1$.

1 child, $\varphi''(s)$ is strictly positive and so φ is *strictly* convex on $(0,1)$). Hence if $m = \varphi'(1_-) > 1$ we must have $\varphi(s_0) < s_0$ for some $s_0 < 1$ and therefore the curves $y = s$ and $y = \varphi(s)$ must cross at some point strictly in $(0,1)$ (recall that $\varphi(0) = p_0 > 0$). A completely rigorous way to justify this is to define $f(s) := \varphi(s) - s$, which is continuous on $[0, s_0]$ with $f(s_0) < 0$ and $f(0) > 0$, so by the intermediate value theorem satisfies $f(s) = 0$, i.e. $\varphi(s) = s$, for some $s \in (0, s_0)$. Conversely, suppose that $m \le 1$ and $p_1 \ne 1-p_0$. Then the condition $m = \varphi'(1_-) \le 1$ together with strict convexity implies that $\varphi(s) - s$ is strictly decreasing on $[0,1]$, from p_0 at 0 to 0 at 1, and therefore cannot be 0 for any $s < 1$. The case $p_1 = 1 - p_0$ give $\varphi(s) = p_0 + p_1 s$ and it is immediate that the smallest solution of $\varphi(s) = s$ in $[0,1]$ is at 1. This completes the proof. □

Corollary 2.8. *If $s \in [0,1)$, we have $\varphi_n(s) \equiv \mathsf{E}\big(s^{Z_n}\big) \to \rho \in (0,1]$ as $n \to \infty$.*

Remark 2.9. *As a result, the distribution of Z_n converges to that of Z_∞, where $\mathsf{P}(Z_\infty = 0) = \rho$ and $\mathsf{P}(Z_\infty = \infty) = 1 - \rho$.*

Exercise 2.10. *For a branching process with generating function $\varphi(s) = as^2 + bs + c$, where $a > 0$, $b > 0$, $c > 0$, $\varphi(1) = 1$, compute the extinction probability ρ and give the condition for sure extinction. Can you interpret your results?*

Exercise 2.11. *Let $(Z_n)_{n\ge0}$ be a branching process with generating function $\varphi(s) \equiv \mathsf{E}s^{Z_1}$ satisfying $0 < \varphi(0) < 1$. Let*

$$\bar\varphi_n(u) \stackrel{\mathsf{def}}{=} \mathsf{E}\big(u^{\bar Z_n}\big)$$

be the generating function of

$$\bar Z_n = \sum_{k=0}^{n} Z_k,$$

the total population size up to time n.

(a) Show that $\bar\varphi_{n+1}(u) = u\,\varphi\big(\bar\varphi_n(u)\big)$ for all $n \ge 0$ and $u \ge 0$.

(b) If $u \in (0,1)$, show that $\bar\varphi_n(u) \to \bar\varphi(u) \stackrel{\mathsf{def}}{=} \mathsf{E}\big(u^{\bar Z}\big)$ as $n \to \infty$, where $\bar Z$ is the total population size, $\sum_{k\ge0} Z_k$, of $(Z_n)_{n\ge0}$. Show that the limiting generating function $\bar\varphi(u)$ is given by the smallest positive solution s to the equation $s = u\varphi(s)$.

(c) Let the process $(Z_n)_{n\ge0}$ be subcritical ($\varphi'(1_-) < 1$) with offspring distribution having exponential tails ($\varphi(s) < \infty$ for some $s > 1$). Show that for some $u > 1$ the equation $s = u\varphi(s)$ has positive solutions s, the smallest of which coincides with $\bar\varphi(u) = \mathsf{E}\big(u^{\bar Z}\big)$.

(d) Let the process $(Z_n)_{n\ge0}$ be supercritical ($\mathsf{E}Z_1 > 1$) with $0 < \mathsf{P}(Z_1 = 0) < 1$ and let $u > 1$ be such that the equation $s = u\varphi(s)$ has positive solutions s. Show that $\bar\varphi_n(u) \to \infty$, in agreement with the fact that with positive probability the process $(Z_n)_{n\ge0}$ survives forever, $\mathsf{P}(\bar Z = \infty) > 0$.

(e) *In the setting of part d), let* $\hat\varphi_n(u) \stackrel{\mathsf{def}}{=} \mathsf{E}\big(u^{\bar Z_n}\mathsf{1}_{Z_n=0}\big)$*, the generating function of the total population on the event that the process* $(Z_n)_{n\ge0}$ *dies out by time* n*. Show that* $\hat\varphi_{n+1}(u) = u\varphi\big(\hat\varphi_n(u)\big)$ *for all* $n\ge0$ *and* $u\ge0$*. Deduce that for each* $u>1$ *such that the fixed point equation* $s=u\varphi(s)$ *has positive solutions* s*, we have* $\hat\varphi_n(u)\to\mathsf{E}\big(u^{\bar Z}\mathsf{1}_{\bar Z<\infty}\big)$*, where the latter coincides with the smallest positive* s *satisfying* $s=u\varphi(s)$*.*

We now turn to classification of states for the Markov chain Z_n in $\mathbb{Z}^+$. Of course, since 0 is an absorbing state, it is recurrent.

Lemma 2.12. *If* $p_1=\mathsf{P}(Z_1=1)\neq1$*, then every state* $k\in\mathbb{N}$ *is transient. As a result,*

$$\mathsf{P}(Z_n\to\infty)=1-\mathsf{P}(Z_n\to0)=1-\rho\,.$$

Proof. We first show that every $k\in\mathbb{N}$ is transient. If $p_0=0$, then Z_n is a non-decreasing Markov chain (ie., $Z_{n+1}\ge Z_n$), so that for every $k\in\mathbb{N}$ the first passage probability f_{kk} satisfies

$$f_{kk}=\mathsf{P}\big(Z_{n+1}=k\mid Z_n=k\big)=\mathsf{P}_k(Z_1=k)=(p_1)^k<1\,.$$

On the other hand, for $p_0\in(0,1]$ we have

$$f_{kk}\le\mathsf{P}\big(Z_{n+1}\neq0\mid Z_n=k\big)=\mathsf{P}_k(Z_1\neq0)=1-\mathsf{P}_k(Z_1=0)=1-(p_0)^k<1\,.$$

This means that

$$\mathsf{P}(Z_n=k\text{ i.o.})=\lim_{m\to\infty}\mathsf{P}(Z_n\text{ returns to }k\text{ at least }m\text{ times})\le\lim_{m\to\infty}f_{kk}^{m-1}=0$$

and the state k is transient.

Fix arbitrary $K>0$. Since the states 1, 2, ..., K are transient, we see that $\mathsf{P}\big(\{Z_n=0\}\cup\{Z_n>K\}\big)\to1$ as $n\to\infty$ and therefore

$$\mathsf{P}\big(Z_n\to0\text{ or }Z_n\to\infty\big)=1\,.$$

As the LHS above equals $\mathsf{P}(Z_n\to0)+\mathsf{P}(Z_n\to\infty)$, the result follows from the observation that $\mathsf{P}(Z_n\to0)\equiv\mathsf{P}(\mathcal{E})=\rho$. □

Exercise 2.13. *For a supercritical branching process* $(Z_n)_{n\ge0}$*, let* $T_0=\min\{n\ge0:Z_n=0\}$ *be its extinction time and let* $\rho=\mathsf{P}(T_0<\infty)>0$ *be its extinction probability. Define* $(\widehat Z_n)_{\ge0}$ *as* Z_n *conditioned on extinction, ie.,* $\widehat Z_n=\big(Z_n\mid T_0<\infty\big)$*.*

(a) *Show that the transition probabilities* $\hat p_{xy}$ *of* $(\widehat Z_n)_{n\ge0}$ *and the transition probabilities* p_{xy} *of the original process* $(Z_n)_{n\ge0}$ *are related via* $\hat p_{xy}=p_{xy}\rho^{y-x}$*,* $x,y\ge0$*.*

(b) *Deduce that the generating functions* $\widehat\varphi(s)\equiv\mathsf{E}_1\big[s^{\widehat Z_1}\big]$ *and* $\varphi(s)\equiv\mathsf{E}_1\big[s^{Z_1}\big]$ *are related via*[10] $\widehat\varphi(s)=\frac{1}{\rho}\varphi(\rho s)$*,* $0\le s\le1$*.*

[10] Geometrically, the graph of this generating function is a rescaled version of that of $\varphi(\cdot)$.

(c) *If the offspring distribution of* $(Z_n)_{n\geq 0}$ *is* $\mathsf{Poi}(\lambda)$ *with* $\lambda > 1$*, use the fixed point equation* $\rho = e^{\lambda(\rho-1)}$ *to show that* $\widehat{\varphi}(s) = e^{\lambda\rho(s-1)}$*, ie., that the offspring distribution for* $(\widehat{Z}_n)_{n\geq 0}$ *is just* $\mathsf{Poi}(\lambda\rho)$*.*

Exercise 2.14. *Let* $(Z_n)_{n\geq 0}$ *be a supercritical branching process with offspring distribution* $\{p_k\}_{k\geq 0}$*, offspring generating function* $\varphi(s)$ *and extinction probability* $\rho \in [0,1)$*.*

(a) *If* $Z_0 = 1$*, let* $\tilde{p}_k$ *be the probability that conditioned on survival the first generation has exactly* k *individuals with an infinite line of descent. Show that*

$$\tilde{p}_k = \frac{1}{1-\rho}\sum_{n=k}^{\infty} p_n \binom{n}{k} (1-\rho)^k \rho^{n-k}\,.$$

(b) *Let* $(\widetilde{Z}_n)_{n\geq 0}$ *count only those individuals in* $(Z_n)_{n\geq 0}$*, who conditioned on survival have an infinite line of descent. Show that* $(\widetilde{Z}_n)_{n\geq 0}$ *is a branching process with offspring generating function*[10]

$$\widetilde{\varphi}(s) = \frac{1}{1-\rho}\Big(\varphi\big((1-\rho)s+\rho\big) - \rho\Big)\,.$$

Exercise 2.15. *Let* $(Z_n)_{n\geq 0}$ *be a subcritical branching process whose generating function* $\varphi(s) = \mathsf{E}(s^{Z_1})$ *is finite for some* $s > 1$*, ie., the offspring distribution has finite exponential moments in a neighbourhood of the origin.*

(a) *Using the result of Exercise 2.11 or otherwise, show that the total population size* $\bar{Z} = \sum_{k\geq 0} Z_k$ *satisfies* $\mathsf{E}\big(u^{\bar{Z}}\big) < \infty$ *for some* $u > 1$*.*

(b) *Suppose that for each* $1 \leq i \leq \bar{Z}$*, individual* i *produces wealth of size* W_i*, where* W_i *are independent random variables with common distribution satisfying* $\mathsf{E}\big(s^W\big) < \infty$ *for some* $s > 1$*. Show that for some* $u > 1$ *we have* $\mathsf{E}\big(u^{\overline{W}}\big) < \infty$*, where* $\overline{W} = W_1 + \cdots + W_{\bar{Z}}$ *is the total wealth generated by* $(Z_n)_{n\geq 0}$*.*

2.2 Critical case $m = 1$

The following example is one of very few for which the computation in the critical case $m = \mathsf{E}(Z_1) = 1$ can be done explicitly.

Example 2.16. *Consider the so-called linear-fractional case, where the offspring distribution is given by* $p_j = 2^{-(j+1)}$*,* $j \geq 0$*. Then the offspring generating function is* $\varphi(s) = \sum_{j\geq 0} s^j/2^{j+1} = (2-s)^{-1}$ *and a straightforward induction gives* (check this!)

$$\varphi_k(s) = \frac{k-(k-1)s}{(k+1)-ks} = \frac{k}{k+1} + \frac{1}{k(k+1)}\sum_{m\geq 1}\Big(\frac{ks}{k+1}\Big)^m\,,$$

so that $\mathsf{P}(Z_k = 0) = \varphi_k(0) = k/(k+1)$, $\mathsf{P}(Z_k > 0) = 1/(k+1)$, *and*

$$\mathsf{P}\big(Z_k = m \mid Z_k > 0\big) = \frac{1}{k+1}\Big(\frac{k}{k+1}\Big)^{m-1}, \qquad m \ge 1,$$

ie., $\big(Z_k \mid Z_k > 0\big)$ *has geometric distribution with success probability* $1/(k+1)$.

Remark 2.17. *For each* $k \ge 0$*, by the partition theorem,*

$$1 = \mathsf{E}(Z_k) = \mathsf{E}\big(Z_k \mid Z_k > 0\big)\,\mathsf{P}(Z_k > 0) + \mathsf{E}\big(Z_k \mid Z_k = 0\big)\,\mathsf{P}(Z_k = 0),$$

so that in the previous example we have

$$\mathsf{E}\big(Z_k \mid Z_k > 0\big) = \frac{1}{\mathsf{P}(Z_k > 0)} = k+1,$$

ie., conditional on survival, the average generation size grows linearly with time.

The following example is known as the general linear-fractional case:

Exercise 2.18. *For fixed* $b > 0$ *and* $p \in (0,1)$*, consider a branching process with offspring distribution* $p_j = b\,p^{j-1}$*,* $j \ge 1$*, and* $p_0 = 1 - \sum_{j\ge1} p_j$*.*

(a) Show that for $b \in (0, 1-p)$ *the distribution above is well defined; find the corresponding* p_0*, and show that*

$$\varphi(s) = \frac{1-b-p}{1-p} + \frac{bs}{1-ps};$$

(b) Find b *for which the branching process is critical and show that then*

$$\varphi_k(s) = \mathsf{E}\big(s^{Z_k}\big) = \frac{kp - (kp+p-1)s}{(1-p+kp) - kps};$$

(c) Deduce that $\big(Z_k \mid Z_k > 0\big)$ *is geometrically distributed with parameter* $\frac{1-p}{kp+1-p}$*.*

Straightforward computer experiments show that a similar linear growth of $\mathsf{E}\big(Z_k \mid Z_k > 0\big)$ takes place for other critical offspring distributions, eg., the one with $\varphi(s) = (1+s^2)/2$.

Theorem 2.19. *If the offspring distribution of the branching process* $(Z_k)_{k\ge0}$ *has mean* $m = 1$ *and finite variance* $\sigma^2 > 0$*, then* $k\,\mathsf{P}(Z_k > 0) \to \frac{2}{\sigma^2}$ *as* $k \to \infty$*; equivalently,*

$$\frac{1}{k}\,\mathsf{E}\big(Z_k \mid Z_k > 0\big) \to \frac{\sigma^2}{2} \qquad \text{as } k \to \infty. \tag{2.4}$$

Remark 2.20. *This general result suggests that, conditional on survival, a general critical branching process exhibits linear intermittent behaviour;*[11] *namely, with small probability (of order* $2/(k\sigma^2)$*) the values of* Z_k *are of order* k*.*

[11] Intermittency follows from the criticality condition, $1 = \mathsf{E}(Z_k \mid Z_k > 0)\mathsf{P}(Z_k > 0)$; it is the linearity which is surprising here!

Our argument is based on the following general fact:[12]

Lemma 2.21. *Let $(y_n)_{n\geq 0}$ be a real-valued sequence. If for some constant a we have $y_{n+1} - y_n \to a$ as $n \to \infty$, then $n^{-1} y_n \to a$ as $n \to \infty$.*

Proof. By changing the variables $y_n \mapsto y'_n = y_n - na$ if necessary, we can and shall assume that $a = 0$. Fix arbitrary $\delta > 0$ and find $K > 0$ such that for $n \geq K$ we have $|y_{n+1} - y_n| \leq \delta$. Decomposing, for $n > K$,
$y_n - y_K = \sum_{j=K}^{n-1} \big(y_{j+1} - y_j\big)$ we deduce that $|y_n - y_K| \leq \delta(n - K)$ so that the claim follows from the estimate

$$\left|\frac{y_n}{n}\right| \leq \left|\frac{y_n - y_K}{n}\right| + \left|\frac{y_K}{n}\right| \leq \delta + \left|\frac{y_K}{n}\right| \leq 2\delta\,,$$

provided n is chosen sufficiently large. $\square$

Proof. (of Theorem 2.19). We only derive the second claim of the theorem, (2.4). By assumptions and Taylor's theorem (here we are also using Theorem 1.16) the offspring generating function φ satisfies

$$1 - \varphi(s) = (1 - s) + \frac{\sigma^2}{2}(1 - s)^2 + R(s)(1 - s)^2 \text{ where } R(s) \to 0 \text{ as } s \uparrow 1.$$

Since $\varphi_n(0) = \mathsf{P}(Z_n = 0) \to 1$ as $n \to \infty$ this means in particular that $1 - \varphi_{n+1}(0) = 1 - \varphi(\varphi_n(0)) = 1 - \varphi_n(0) + (1 - \varphi_n(0))^2(\frac{\sigma^2}{2} + r_n))$ with $r_n := R(\varphi_n(0)) \to 0$ as $n \to \infty$. Setting

$$y_n = \frac{1}{1 - \varphi_n(0)} = \frac{\mathsf{E}(Z_n)}{\mathsf{P}(Z_n > 0)} = \mathsf{E}(Z_n | Z_n > 0)$$

we therefore have

$$y_{n+1} - y_n = \frac{1}{1 - \varphi_n(0)} \frac{1}{1 - (1 - \varphi_n(0))(\sigma^2/2 + r_n)} - 1$$

which we can rewrite as

$$\frac{(1 - \varphi_n(0))(\sigma^2/2 + r_n)}{1 - \varphi_n(0)} \frac{1}{1 - (1 - \varphi_n(0))(\sigma^2/2 + r_n)}$$

for each n. It therefore follows that $y_{n+1} - y_n \to \sigma^2/2$ as $n \to \infty$ and hence

$$\frac{y_n}{n} = \frac{\mathsf{E}(Z_n | Z_n > 0)}{n} \to \frac{\sigma^2}{2}$$

as well (by the Lemma). This completes the proof. $\square$

[12] Compare the result to Cesàro limits of real sequences: if $(a_k)_{k\geq 1}$ is a real-valued sequence, and $s_n = a_1 + \cdots + a_n$ is its nth partial sum, then $\frac{1}{n} s_n$ are called the *Cesàro* averages for the sequence $(a_k)_{k\geq 1}$. Lemma 2.21 claims that if $a_k \to a$ as $k \to \infty$, then the sequence of its Cesàro averages also converges to a. The converse is, of course, false. (Find a counterexample!)

Remark 2.22. *With a bit of extra work* [13] *one can generalize the above proof to show that*

$$\lim_{n\to\infty} n^{-1}\Big(\frac{1}{1-\varphi_n(s)} - \frac{1}{1-s}\Big) = \frac{\sigma^2}{2}$$

for any $s \in [0,1]$ and use this relation to derive the convergence in distribution:

Theorem 2.23. *If $\mathsf{E}Z_1 = 1$ and $\mathsf{Var}(Z_1) = \sigma^2 \in (0,\infty)$, then for every $z \geq 0$ we have*

$$\lim_{k\to\infty} \mathsf{P}\Big(\frac{Z_k}{k} > z \mid Z_k > 0\Big) = \exp\Big\{-\frac{2z}{\sigma^2}\Big\},$$

ie., the distribution of $\big(k^{-1}Z_k \mid Z_k > 0\big)$ is approximately exponential with parameter $2/\sigma^2$.

Remark 2.24. *In the setup of Example 2.16, we have*

$$\mathsf{P}\big(Z_k > m \mid Z_k > 0\big) = \Big(\frac{k}{k+1}\Big)^m = \Big(1 - \frac{1}{k+1}\Big)^m,$$

so that $\mathsf{P}\big(Z_k > kz \mid Z_k > 0\big) \to e^{-z}$ as $k \to \infty$; in other words, for large k the distribution of $\big(k^{-1}Z_k \mid Z_k > 0\big)$ is approximately $\mathsf{Exp}(1)$.

Exercise 2.25. *Let $(Z_n)_{n\geq 0}$ be the critical branching process from Exercise 2.18, namely, the one whose offspring distribution is given by $(p_j)_{j\geq 0}$,*

$$p_j = b\,p^{j-1}, \quad j \geq 1, \qquad p_0 = 1 - \sum_{j\geq 1} p_j,$$

where $b > 0$ and $p \in (0,1)$ are fixed parameters. Show that the result of Theorem 2.23 holds: for every $z \geq 0$

$$\lim_{k\to\infty} \mathsf{P}\Big(\frac{Z_k}{k} > z \mid Z_k > 0\Big) = \exp\Big\{-\frac{2z}{\sigma^2}\Big\},$$

where $\mathsf{Var}(Z_1) = \sigma^2 \in (0,\infty)$.

2.3 Non-homogeneous case

If the offspring distribution changes with time, the previous approach must be modified. Let $\psi_n(u)$ be the generating function of the offspring distribution of a single ancestor in the $(n-1)$st generation,

$$\psi_n(u) = \mathsf{E}\big(u^{Z_n} \mid Z_{n-1} = 1\big)$$

(so in the cases considered up to now, $\psi_n = \varphi$ for every n). Then the generating function $\varphi_n(u) = \mathsf{E}\big(u^{Z_n} \mid Z_0 = 1\big)$ of the population size at time n given a single ancestor at time 0, can be defined recursively as follows:

$$\varphi_0(u) \equiv u, \qquad \varphi_n(u) = \varphi_{n-1}\big(\psi_n(u)\big), \quad \forall n \geq 1.$$

[13] using the fact that every $s \in (0,1)$ satisfies $0 < s < \varphi_k(0) < 1$ for some $k \geq 1$;

If $\mu_n = \mathsf{E}(Z_n \mid Z_{n-1} = 1) = \psi_n'(1)$ denotes the average offspring size in the nth generation given a single ancestor in the previous generation, then

$$m_n \equiv \mathsf{E}\big(Z_n \mid Z_0 = 1\big) = \mu_1 \mu_2 \dots \mu_{n-1} \mu_n \,.$$

It is natural to call the process $(Z_n)_{n\geq 0}$ supercritical if $m_n \to \infty$ and subcritical if $m_n \to 0$ as $n \to \infty$.

Exercise 2.26. *A strain of phototrophic bacteria uses light as the main source of energy. As a result individual organisms reproduce with probability mass function $p_0 = 1/4$, $p_1 = 1/4$ and $p_2 = 1/2$ per unit of time in light environment, and with probability mass function $p_0 = 1-p$, $p_1 = p$ (with some $p > 0$) per unit of time in dark environment. A colony of such bacteria is grown in a laboratory, with alternating light and dark unit time intervals.*

a) Model this experiment as a time non-homogeneous branching process $(Z_n)_{n\geq 0}$ and describe the generating function of the population size at the end of the nth interval.

b) Characterise all values of p for which the branching process Z_n is subcritical and for which it is supercritical.

c) Let $(D_k)_{k\geq 0}$ be the original process observed at the end of each even interval, $D_k \stackrel{\mathsf{def}}{=} Z_{2k}$. Find the generating function of $(D_k)_{k\geq 0}$ and derive the condition for sure extinction. Compare your result with that of part b).