Trilogarithms and volumes of hyperbolic 5-manifolds

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"Polylogarithms, Cluster Algebras, and Scattering Amplitudes"

Brin MRC, 11–15 September 2023

Non-euclidean volume

Let X_K^n be a standard geometric space of constant curvature $K \in \{-1, 0, +1\}$ with its distance function d_X.

- For $K = +1$ and $\mathbb{S}^n \subset \mathbb{R}^{n+1}$: $d_{\mathbb{S}}(x, y) = \arccos(x \cdot y)$
- For $K = -1$ and $\mathbb{H}^n \subset \mathbb{R}^n_+$ in the upper half space:

$$d_{\mathbb{H}}(x, y) = \log \frac{y_n}{x_n} \quad \text{for}$$

$$x = (0, \ldots, 0, x_n)\,,\ y = (0, \ldots, 0, y_n)\,,\ y_n \geq x_n$$

Consequences.

- For $K \neq 0$, distances are related to $\mathrm{Li}_1(z) = -\log(1 - z)$
- The hyperbolic volume element equals

$$d\,\mathrm{vol}_n = \frac{dx_1 \cdots dx_n}{x_n^n}$$

Non-euclidean polyhedra and Schläfli's volume differential

Theorem (L. Schläfli; H. Kneser; J. Milnor)

For a non-euclidean n-simplex $S \subset X^n_K$ *with dihedral angles* α_F *at* $(n-2)$*-dimensional faces* $F \subset S$*, one has*

$$d\,\mathrm{vol}_n(S) = \frac{K}{n-1} \sum_F \mathrm{vol}_{n-2}(F)\, d\,\alpha_F \quad ,$$

where $\mathrm{vol}_0(S) := 1$.

First implications.

- The non-euclidean volume problem is subdivided according to the dimension parity
- Excess formula resp. defect formula for $n = 2$ and $K \neq 0$
- Schläfli's simplex reduction formula for $\mathrm{vol}_{2m}(S) \subset \mathbb{S}^{2m}$

Polylogarithms and related functions

Classical polylogarithms. (Leibniz, Johann Bernoulli)

$$\mathrm{Li}_n(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^n} \quad , \quad z \in \mathbb{C} \quad \text{with} \quad |z| < 1$$

$$\mathrm{Li}_1(z) = -\log(1-z) \quad ; \quad \mathrm{Li}_n(z) = \int_0^z \mathrm{Li}_{n-1}(t)\, d\log t$$

Lobachevsky function. Let $\alpha \in \mathbb{R}$.

$$Л_2(\alpha) = \frac{1}{2}\,\Im\,\mathrm{Li}_2(e^{2i\alpha}) = \frac{1}{2}\sum_{r=1}^{\infty} \frac{\sin(2r\alpha)}{r^2}$$

Some modified dilogarithms (Bloch–Wigner,...)

$$\mathscr{L}_1(z) = \Re\,\log z = \log|z|$$

$$D(z) = \mathscr{L}_2(z) = \Im\,\{\,\mathrm{Li}_2(z) - \mathrm{Li}_1(z)\log|z|\,\}$$

$$\sum_{k=1}^{5} (-1)^k\,\mathscr{L}_2(r_2(z_1,\ldots,\widehat{z_k},\ldots,z_5)) = 0 \quad \text{for} \quad z_i \neq z_j$$

5-term relation of Spence-Abel

The context of hyperbolic scissors congruences

For the scissors congruence group $\mathscr{P}(X^n_K)$ of polyhedra in X^n_K:

The (classes of) **orthoschemes** $[\alpha_1, \ldots, \alpha_n]$ generate $\mathscr{P}(X^n_K)$.

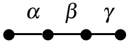
By results of Sah and Debrunner:

- For $n \geq 2$, the image of $\mathscr{P}(\mathbb{H}^n)$ in $\mathscr{P}(\overline{\mathbb{H}^n})$ is generated by **1-asymptotic** orthoschemes
- For $n \geq 3$ and **odd**, $\mathscr{P}(\overline{\mathbb{H}^n})$ is generated by **2-asymptotic** orthoschemes

Consequence. In **odd** dimensions, it suffices to solve the volume problem for **2-asymptotic orthoschemes**

Hyperbolic volume in three dimensions

Lobachevsky derived a closed volume formula for the generating orthoschemes in $\mathscr{P}(\mathbb{H}^3)$ in terms of the dihedral angle parameters α, β, γ by introducing a new function, the *Lobachevsky function* $\text{Л}_2(\omega)$ as introduced earlier.

For an orthoscheme $R = [\alpha, \beta, \gamma] \subset \mathbb{H}^3$ with graph 

$$\text{vol}_3(R) = \frac{1}{4}\Big\{\, \text{Л}_2(\alpha+\theta) - \text{Л}_2(\alpha-\theta) + \text{Л}_2(\frac{\pi}{2}+\beta-\theta) +$$

$$+\text{Л}_2(\frac{\pi}{2}-\beta-\theta) + \text{Л}_2(\gamma+\theta) - \text{Л}_2(\gamma+\theta)\,\Big\} \quad ,$$

$$0 \le \theta = \arctan \frac{\sqrt{\cos^2\beta - \sin^2\alpha \sin^2\gamma}}{\cos\alpha\cos\gamma} \le \frac{\pi}{2}$$

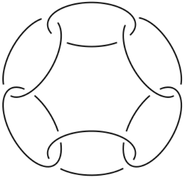
Polyhedral dissection and functional equations

Different cutting and pasting procedures applied to a polyhedron $P \subset \mathbb{H}^3$ lead to functional equations of dilogarithm functions.

Example. The hyperbolic $2n$-chain link complement manifold $M_n = \mathbb{S}^3 \setminus \mathscr{D}_{2n}$ can be built on **different** polyhedral objects providing the functional equation

$$4\left\{Л_2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+Л_2\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\right\} = Л_2\left(\frac{\pi}{2}+\alpha-\theta\right)+Л_2\left(\frac{\pi}{2}-\alpha-\theta\right)+4\,Л_2(\theta)+2\,Л_2\left(\frac{\pi}{2}-\theta\right),$$

where $\alpha, \theta \in [0, \frac{\pi}{2})$ are such that $\tan\theta = \cos\alpha$.



Milnor's volume formula

Let $S_\infty(z) \subset \mathbb{H}^3 = \mathbb{C} \times \{t > 0\}$ be an ideal tetrahedron with vertices $\infty, 0, 1$ and $z \in \mathbb{C} \setminus \mathbb{R}$:

$$\mathrm{vol}_3(S_\infty(z)) = D(z) = Л_2(\alpha) + Л_2(\beta) + Л_2(\gamma), \quad \text{where}$$

$$\alpha = \arg z \quad , \quad \beta = \arg(1 - 1/z) \quad , \quad \gamma = \pi - (\alpha + \beta)$$

Milnor's Conjecture. Let $\{\theta_j\} \subset \mathbb{Q}\pi$. Then, very $\mathbb{Q}$-linear relation

$$\sum_j q_j Л_2(\theta_j) = 0$$

is a consequence of the relations

$$Л_2(x + \pi) = Л_2(x) \quad , \quad Л_2(-x) = Л_2(x)$$

$$Л_2(nx) = n \sum_{k \bmod n} Л_2(x + \frac{k\pi}{n}) \qquad \text{distribution law}$$

The generalised 3rd Problem of Hilbert

Problem : Two polyhedra $P_1, P_2 \subset X^3_K$ are scissors congruent if and only if their volumes and their Dehn invariants are equal, i.e.

$$\mathrm{vol}_3(P_1) = \mathrm{vol}_3(P_2) \quad , \quad \mathrm{Dehn}(P_1) = \mathrm{Dehn}(P_2)$$

where

$$\mathrm{Dehn}(P) = \sum_F \mathrm{vol}_1(F) \otimes \alpha_F \in \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z}$$

Theorem (J. Dupont-H. Sah)

Let $S_\infty(z_n) = (\infty, 0, 1, z_n) \subset \mathbb{H}^3$ *with* $z_n = e^{2\pi i/n}$ *for* $n \geq 7$.
Let $\theta \in]\frac{1}{6}, \frac{1}{2}[$ *so that* $Л_2(\theta\pi) = Л_2(\pi/n) = \frac{1}{2}\,\mathrm{vol}_3(S_\infty(z_n))$.

Then, there is the following alternative :

- $\theta \in \mathbb{R} - \mathbb{Q}$, *i.e.* $\mathrm{Dehn}(S_\infty(\theta)) \neq 0$, *and hence there is a pair of ideal tetrahedra with* equal volume *and* different Dehn-values
- $\theta \in \mathbb{Q}$, *i.e. Milnor's Conjecture is FALSE*

Triangulated hyperbolic 3-manifolds

Theorem (W. Thurston, W. Neumann–D. Zagier,...)

Let M be an oriented hyperbolic 3-manifold of finite volume. Then, there are finitely many algebraic numbers z_i, $i \in I$, satisfying

$$\sum_{i \in I} z_i \wedge (1 - z_i) = 0 \quad \textit{in} \quad \bigwedge{}^2 \mathbb{Q}^\times \qquad (\star)$$

such that $\qquad \mathrm{vol}_3(M) = \sum_{i \in I} \mathscr{L}_2(z_i)\,.$

Interpretation of $(\star)$. The Dehn invariant Dehn(P) can be extended to a Dehn invariant $\Delta(P)$ for non-compact polyhedra $P \subset \mathbb{H}^3$ (cut off ideal vertices by means of small horospheres and measure then edge length).

Example. $\Delta(S_\infty(z)) = 2\{\log|1-z| \otimes \arg z - \log|z| \otimes \arg(1-z)\}$

$$= z \wedge (1-z) - \bar{z} \wedge (1 - \bar{z}) \qquad \text{where}$$

$$\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/2\pi\mathbb{Z} \cong \bigwedge{}^2 (\mathbb{C}^\times)^- \,:\, r \otimes \theta \bmod 2\pi \mapsto -e^r \wedge e^{i\theta}$$

Higher Lobachevsky functions

$$Л_{2k}(\alpha) = \frac{1}{2^{2k-1}} \Im\big(\mathrm{Li}_{2k}(e^{2i\alpha})\big) = \frac{1}{2^{2k-1}} \sum_{r=1}^{\infty} \frac{\sin(2r\alpha)}{r^{2k}}$$

$$Л_{2k+1}(\alpha) = \frac{1}{2^{2k}} \Re\big(\mathrm{Li}_{2k+1}(e^{2i\alpha})\big) = \frac{1}{2^{2k}} \sum_{r=1}^{\infty} \frac{\cos(2r\alpha)}{r^{2k+1}}$$

- $Л_{2k}(\alpha) = \int_0^\alpha Л_{2k-1}(t)dt$
- $Л_{2k+1}(\alpha) = \frac{1}{2^{2k}} \zeta(2k+1) - \int_0^\alpha Л_{2k}(t)dt$

- $Л_k(\alpha)$ is π-periodic, even (odd) for k odd (even) and satisfies

$$\frac{1}{m^{k-1}} Л_k(m\alpha) = \sum_{r=0}^{m-1} Л_k\left(\alpha + \frac{r\pi}{m}\right)$$

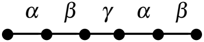
- $Л_3(0) = \frac{1}{4}\zeta(3) \quad , \quad Л_3(\frac{\pi}{6}) = \frac{1}{12}\zeta(3) \quad , \quad Л_3(\frac{\pi}{2}) = -\frac{3}{16}\zeta(3)$

$$Л_3(\frac{\pi}{5}) + Л_3(\frac{2\pi}{5}) = -\frac{3}{25}\zeta(3)$$

A few volume formulae in five dimensions

Theorem (K, 1992)

For a 2-asymptotic orthoscheme $R_\infty \subset \mathbb{H}^5$



with $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, *the volume is given by*

$$\mathrm{vol}_5(R_\infty) = \frac{1}{4}\{\text{Л}_3(\alpha) + \text{Л}_3(\beta) - \frac{1}{2}\text{Л}_3(\frac{\pi}{2} - \gamma)\} -$$

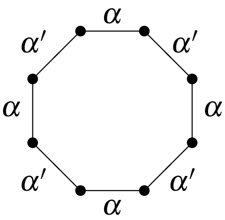
$$-\frac{1}{16}\{\text{Л}_3(\frac{\pi}{2} + \alpha + \beta) + \text{Л}_3(\frac{\pi}{2} - \alpha + \beta)\} + \frac{3}{64}\,\zeta(3)$$

A result of Hild, 2007. The (unique) minimal volume non-compact hyperbolic 5-orbifold $\mathbb{H}^5/\Gamma_*$ is of volume

$$\mathrm{vol}_5([3,4,3,3,3]) = \frac{7}{46080}\,\zeta(3)$$

The volume of a doubly truncated 5-orthoscheme

$\Omega(\alpha)$:



, $\quad \alpha' := \frac{\pi}{2} - \alpha$

Proposition (K, 1995)

The volume of a doubly-truncated orthoscheme in $\mathbb{H}^5$ with cyclic graph $\Omega(\alpha)$ as given above equals

$$\mathrm{vol}_5(\Omega(\alpha)) = \frac{1}{32}\,\zeta(3) - \frac{1}{2}\left\{ \text{Л}_3(\alpha) + \text{Л}_3\left(\frac{\pi}{2} - \alpha\right) \right\}.$$

Example. The Coxeter polyhedron $\Omega(\frac{\pi}{3})$ has volume $\frac{13}{288}\,\zeta(3)$

An application and the value $\zeta(3)$

The covolume of the hybrid quaternionic modular group $\mathrm{PSL}_\Delta(2,\mathbb{H}\mathrm{yb}) = \mathrm{PSL}_\Delta(2,\mathbb{Z}[\omega,j])$ with $\omega = -\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i$ acting discretely on $\mathbb{H}^5$ can be expressed (by using a particular dissection of a fundamental polyhedron) as follows.

Theorem (K, 2018)

$$\mathrm{covol}_5(\mathrm{PSL}_\Delta(2,\mathbb{H}\mathrm{yb})) = \frac{13}{180}\,\zeta(3) = 32\cdot \mathrm{vol}_5(\Omega(\frac{\pi}{3}))$$

By means of Schläfli's volume differential, one deduces the identity

Corollary

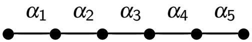
$$\zeta(3) = \frac{360}{13}\left[\,\frac{\pi}{4}\,Л_2(\frac{\pi}{3}) + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left\{\,Л_2(\frac{\pi}{6}+\beta(t)) + Л_2(\frac{\pi}{6}-\beta(t)\,\right\} dt\,\right],$$

where $\quad \cos\beta(t) = \frac{\sin t}{\sqrt{4\sin^2 t - 1}}$

The generic case

Theorem (K, 1995)

Let $R \subset \mathbb{H}^5$ *be a 2-asymptotic orthoscheme with graph* $\Sigma(R)$



Put $\lambda = \tan\Theta = \dfrac{|\det\Sigma(R)|^{1/2}}{\cos\alpha_1\cos\alpha_3\cos\alpha_5}$, $0 \le \Theta \le \dfrac{\pi}{2}$, *and*

$$0 \le \alpha_0 \le \frac{\pi}{2} \text{ so that } \tan\alpha_0 = \cot\Theta\tan\alpha_3 \text{ . Then,}$$

Then, $\mathrm{vol}_5(R) =$

$$-\tfrac{1}{8}\{I(\lambda^{-1},0;\alpha_1) + \tfrac{1}{2}I(\lambda,0;\alpha_2) + \tfrac{1}{2}I(\lambda,0;\alpha_4) + I(\lambda^{-1},0;\alpha_5) - \\ -I(\lambda^{-1},0;\alpha_0')\}$$

$$+\tfrac{1}{32}\{I(\lambda,-(\tfrac{\pi}{2}+\alpha_1);\tfrac{\pi}{2}+\alpha_1+\alpha_2) + I(\lambda,-(\tfrac{\pi}{2}-\alpha_1);\tfrac{\pi}{2}-\alpha_1+\alpha_2) \\ -I(\lambda,-(\tfrac{\pi}{2}+\alpha_1);\pi+\alpha_1) - I(\lambda,-(\tfrac{\pi}{2}-\alpha_1);\pi-\alpha_1) \\ -I(\lambda,-(\tfrac{\pi}{2}+\alpha_5);\pi+\alpha_5) - I(\lambda,-(\pi 2-\alpha_5);\pi-\alpha_5) \\ +I(\lambda,-(\tfrac{\pi}{2}+\alpha_5);\tfrac{\pi}{2}+\alpha_5+\alpha_4) + I(\lambda,-(\tfrac{\pi}{2}-\alpha_5);\tfrac{\pi}{2}-\alpha_5+\alpha_4)\}$$

About the trilogarithmic function $I(a,b;x)$

For $a, b \in \mathbb{R}$,

$$I(a,b;x) = \int_{\frac{\pi}{2}}^{x} \text{Л}_2(y)\, d\arctan(a\tan(b+y))$$

$$I(1,b;x) = -\text{Л}_3(x) - \frac{3}{16}\zeta(3)$$

$I(a,b;x)$ is closely related to the integral

$$J(a,b,c;z) = \int_0^z \log(1+at)\log(1+bt)\, d\log(1+ct)$$

which can be expressed in terms of polylogarithms of orders ≤ 3.

Question. Simple relations of I, J to modified trilogarithms ?

Prasad's formula for arithmetic lattices

Prasad's volume formula was exploited by M. Belolipetsky ($n \geq 4$ even) and Emery ($n \geq 5$ odd). Here are some of Emery's results for weak variants of arithmetic hyperbolic lattices in $\mathrm{PO}(5,1)$.

Theorem (V. Emery, 2016)

Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^5)$ be a non-uniform quasi-arithmetic lattice with associated field extension $\ell/\mathbb{Q}$.

1. *If $\ell = \mathbb{Q}$, then* $\quad \mathrm{vol}_5(\mathbb{H}^5/\Gamma) \in \zeta(3)\cdot\mathbb{Q}^\times$;
2. *otherwise,* $\quad \mathrm{vol}_5(\mathbb{H}^5/\Gamma) \in |\mathrm{D}_\ell|^{5/2}\cdot\frac{\zeta_\ell(3)}{\zeta(3)}\cdot\mathbb{Q}^\times$.

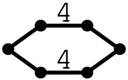
Theorem (Emery-O. Mila, 2021)

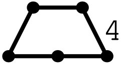
Let $\Gamma \subset \mathrm{Isom}(\mathbb{H}^5)$ be a pseudo-arithmetic lattice of the 1st type, and $\{\Gamma_{\mathrm{i}}\}$ a set of arithmetic lattices subordinated to the ambient group of Γ. Then, there are $\beta_{\mathrm{i}} \in \mathbb{Q}$ such that

$$\mathrm{vol}_5(\mathbb{H}^5/\Gamma) = \sum_{\mathrm{i}} \beta_{\mathrm{i}}\, \mathrm{vol}_5(\mathbb{H}^5/\Gamma_{\mathrm{i}}) \, .$$

About the non-arithmetic hyperbolic Coxeter 5-simplex

The hyperbolic Coxeter 5-simplex groups Δ_{44} and Δ_4 are non-cocompact lattices. The group Δ_{44} is arithmetic while the group Δ_4 is not.





For their covolumes, the integration of Schläfli's volume differential yields (K, 1999)

$$\mathrm{vol}_5(\mathbb{H}^5/\Delta_{44}) = \frac{7}{288}\zeta(3)\,;$$

$$\mathrm{vol}_5(\mathbb{H}^5/\Delta_{4}) = \frac{7}{288}\zeta(3) - \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \mathrm{vol}_3(F(t))\,dt \qquad (\star)$$

$$\approx 0.007573474422\ldots$$

Comparison with the result of Emery-Mila

The Coxeter group Δ_4 is not arithmetic but *pseudo-arithmetic of the 1st type* with 2 subordinated arithmetic lattices $\Gamma_i = \mathrm{PO}_{f_i}(\mathbb{Z})\,,\, i = 0, 1\,,$ where

$$f_0 = -x_0^2 + x_1^2 + \cdots + x_5^2 \quad , \quad f_1 = -x_0^2 + x_1^2 + \cdots + 2x_5^2 \quad \text{and}$$

$$\mathrm{vol}_5(\mathbb{H}^5/\Gamma_0) \in \zeta(3) \cdot \mathbb{Q}^\times \quad , \quad \mathrm{vol}_5(\mathbb{H}^5/\Gamma_1) \in \sqrt{2}\,\mathrm{L}(\chi_8, 3) \cdot \mathbb{Q}^\times$$

Numerical approximation (up to 160 digits; S. Tschantz)

$$\mathrm{vol}_5(\mathbb{H}^5/\Delta_4) \approx \frac{73}{2^9 3^2 5}\zeta(3) + \frac{1}{2^3 3^2 5}\sqrt{2}\,\mathrm{L}(\chi_8, 3) \quad (\star)$$
$$= 0.00757347442200786763497722\ldots$$

Goncharov's structural result part I

Predicted by Zagier's Conjecture, to express $L_{\ell_0/k_0}(3)$, for example, as a sum of the **modified trilogarithms** $\mathcal{L}_3$ evaluated at integers of k_0, is the following result.

Theorem (A. Goncharov, 1998)

Let M *be an oriented hyperbolic 5-manifold of finite volume. Then, there are finitely many* $z_i \in \overline{\mathbb{Q}}, i \in I$, *satisfying*

$$\sum_{i \in I} \{z_i\} \otimes z_i = 0 \quad \textit{in} \quad G(\overline{\mathbb{Q}}) \otimes \overline{\mathbb{Q}}^{\times} \quad \textit{such that}$$

$$\mathrm{vol}_5(M) = q \sum_{i \in I} \mathcal{L}_3(z_i) \quad \textit{for some } q \in \mathbb{Q}^{\times}$$

Here, $\{x\}$ *is an element of the following group* $G(F)$ *...*

Goncharov's structural result - part II

Let F be a number field. Then,

$$G(F) = \frac{\mathbb{Z}[P_1(F)]}{< \sum_{k=1}^{5} (-1)^k \, [r_2(x_1, \ldots, \widehat{x_k}, \ldots, x_5)], [0], [\infty] \mid x_i \neq x_j >}$$

$$\mathscr{L}_3(z) = \Re \{ \operatorname{Li}_3(z) - \operatorname{Li}_2(z) \log |z| + \frac{1}{3} \operatorname{Li}_1(z) \log^2 |z| \}$$

Problem. Find an example with explicit q and z_i's such that

$$\mathrm{vol}_5(M) = q \sum_{i \in I} \mathscr{L}_3(z_i) \quad \text{for some } q \in \mathbb{Q}^\times$$

Zagier's Conjecture and Goncharov's result

Let F be a number field of degree d and of discriminant D_F with r_1 real and $2r_2$ non-real embeddings $\sigma_j : F \hookrightarrow \mathbb{C}$. Let ζ_F be the zeta function of F, and put

$$d_n := \begin{cases} r_1 + r_2 & \text{if } n \text{ is odd} \\ r_2 & \text{if } n \text{ is even} \end{cases}$$

Zagier's Conjecture. *For $n \geq 2$, there is $q_n \in \mathbb{Q}^\times$ such that*

$$\zeta_F(n) = q_n \, \pi^{n(d-d_n)} \, D_F^{-1/2} \, R_n \quad , \quad \textit{where}$$

R_n is a sum of $\mathscr{L}_n$-values taken at elements and their conjugates of F.

Example. $\zeta_{\mathbb{Q}(\sqrt{5})}(3) = \frac{24}{25\sqrt{5}} \, \mathscr{L}_3(1) \left(\mathscr{L}_3(\frac{1+\sqrt{5}}{2}) - \mathscr{L}_3(\frac{1-\sqrt{5}}{2}) \right)$

Theorem (A. Goncharov, 1995)

Let F be a number field with r_1 real and $2r_2$ non-real embeddings $\sigma_j : F \hookrightarrow \mathbb{C}$ such that $\sigma_{r_1+k} = \overline{\sigma_{r_1+r_2+k}}$ as above. Then, there are certain algebraic numbers $\alpha_1, \ldots, \alpha_{r_1+r_2}$ such that

$$\zeta_F(3) = \pi^{3r_2} \, D_F^{-1/2} \det\left(| \mathscr{L}_3(\sigma_j(\alpha_k)) | \right) \quad (1 \leq i,k \leq r_1+r_2)$$

Open problems

- For $n = 7$, find a closed volume formula for an infinite family of polyhedra in $\mathbb{H}^7$

- Hilbert's 3rd problem for $\mathscr{P}(X_K^3)$ for $K \neq 0$

Pseudo-arithmetic lattices of the 1st type

- An algebraic group G_K is **pseudo-admissible** (over K/k) if $K = k(\sqrt{a_1}, \ldots, \sqrt{a_r})$ is totally real and G is an admissible k-group
- For $n > 3$, a lattice $\Gamma \subset \mathrm{PO}(n,1)$ is **pseudo-arithmetic** (over K/k) if its ambient group is pseudo-admissible
- Let $K = k(\sqrt{a_1}, \ldots, \sqrt{a_r})$ be totally real field and f the diagonal quadratic form in $x_0, \ldots, x_n$ with negative coefficient in x_0:
  For a multi-index $i \in \{0,1\}^r$, put

$$\alpha_i := \sqrt{a_1^{i_1} \cdots a_r^{i_r}} \quad \text{and} \quad f_i := f(x_0, \ldots, x_{n-1}, \alpha_i x_n)$$

- For each $i \in \{0,1\}^r$, choose an arithmetic subgroup $\Gamma_i \subset \mathrm{PO}_{f_i}(k)$ (commensurable with $\mathrm{PO}_{f_i}(\mathscr{O}_k)$)
- Then, the set of of arithmetic lattices $\{\Gamma_i \mid 0 \leq i \leq 2^r\}$ is **subordinated to** $\mathrm{PO}_{f,K}$

A structural result about the volume spectrum $\mathscr{V}_3$

Compare the result of Emery-Mila with the classical result of Borel for hyperbolic 3-manifolds.

Theorem (A. Borel)

*For any number field F with r complex places, there are $v_1, \dots, v_r \in \mathbb{R}$ such that for **any** finite-volume hyperbolic 3-manifold M whose invariant trace field is F, there are r numbers $a_1, \dots, a_r \in \mathbb{Q}$ with*

$$\mathrm{vol}_3(M) = a_1 v_1 + \cdots + a_r v_r$$

Remark. If F is a number field with exactly one complex place, then there is a number $v \in \mathbb{R}$ such that every *arithmetic* hyperbolic 3-orbifold (or 3-manifold) Q whose defining field is k has volume which is *rational* multiple of v. In fact, for $d = [F : \mathbb{Q}]$, one can take

$$v = \frac{\mid D_F \mid^{3/2} \zeta_k(2)}{(4\pi^2)^{d-1}}$$