Alternating Multiple Mixed Values

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Outline

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1. Multiple zeta values (MZVs) and some variants
2. Multiple mixed values (MMVs)
3. Alternating multiple mixed values (AMMVs)
4. Dimension computations
5. Finite AMMVs

Multiple zeta values

Definition.

The multiple zeta values (MZVs) are defined by

$$\zeta(k_1, \dots, k_d) = \sum_{n_1 > \cdots > n_d > 0} \frac{1}{n_1^{k_1} \cdots n_d^{k_d}}$$

when $k_1, \dots, k_d$ are positive integers and $k_1 \geq 2$ (i.e. *admissible*).

- d: depth
- $|(k_1, \dots, k_d)| = k_1 + \cdots + k_d$: weight

Multiple zeta values

A key question.

Find all the $\mathbb{Q}$-linear relations among MZVs.

Example. Euler's identity

We have

$$\zeta(3) = \zeta(2,1)$$

$$\sum_{n\geq 1} \frac{1}{n^3} = \sum_{n>m>0} \frac{1}{n^2 m}.$$

Conjecture.

All the $\mathbb{Q}$-linear relations among MZVs are given by the regularized/extended double shuffle relations (DBSF).

Multiple zeta values

Dimension Conjecture. Zagier 1994

Let $\mathbf{MZV}_0 = \mathbb{Q}$ and let $\mathbf{MZV}_w$ be $\mathbb{Q}$-span of MZVs of weight $w \geq 1$. Then

$$\sum_{w\geq 0} (\dim_{\mathbb{Q}} \mathbf{MZV}_w)\, x^w = \frac{1}{1-x^2-x^3}.$$

Basis Conjecture. Hoffman 1990's

The $\mathbb{Q}$-vector space generated by MZVs of weight $w \geq 1$ has a basis

$$\big\{\zeta(s_1,\dots,s_d) : |\boldsymbol{s}| = w, s_j = 2 \text{ or } 3\big\}. \qquad (\star)$$

Theorem. (F. Brown 2012)

The set $(\star)$ is a generating set.

Euler sums

Definition.

Let $k_1, \ldots, k_d \in \mathbb{N}, \eta_1, \ldots, \eta_d \in \{\pm 1\}$. For $(k_1, \eta_1) \neq (1, 1)$, we define the Euler sums (ES) or alternating MZV (AMZV) by

$$\zeta(k_1, \ldots, k_d; \eta_1, \ldots, \eta_d) = \sum_{n_1 > \cdots > n_d > 0} \frac{\eta_1^{k_1} \cdots \eta_d^{k_d}}{n_1^{k_1} \cdots n_d^{k_d}}.$$

Convention

We put a bar on top of s_j if $\eta_j = -1$.

$$\zeta(\bar{1}) = \zeta(1; -1) = \sum_{n \geq 1} \frac{(-1)^n}{n},$$

$$\zeta(\bar{2}, 1) = \zeta(2, 1; -1, 1) = \sum_{n > m > 0} \frac{(-1)^n}{n^2 m}.$$

Euler sums

Theorem. Deligne and Goncharov 2005.

Let $\mathbf{ES}_0 = \mathbb{Q}$ and let $\mathbf{ES}_w$ be the $\mathbb{Q}$-span of all Euler sums of weight $w \geq 1$. Then $\dim_{\mathbb{Q}} \mathbf{ES}_w \leq F_w$ where

$$\sum_{w \geq 0} F_w \, x^w = \frac{1}{1 - x - x^2}.$$

Example.

We have

$$\begin{aligned} \mathbf{ES}_1 &= \langle \zeta(\bar{1}) \rangle_{\mathbb{Q}}, \\ \mathbf{ES}_2 &= \langle \zeta(\bar{1}, \bar{1}), \zeta(\bar{1}, 1), \zeta(\bar{2}), \zeta(2) \rangle_{\mathbb{Q}}. \end{aligned}$$

But

$$\zeta(2) = -2\zeta(\bar{2}), \quad \zeta(\bar{1}, 1) = \zeta(\bar{1}, \bar{1}) - \zeta(\bar{2}).$$

Multiple t-values (MtVs), Hoffman 2019

Definition.

For all admissible $(k_1, \dots, k_d) \in \mathbb{N}^d$, the multiple t-values (MtV) are defined by

$$t(k_1, \dots, k_d) := \sum_{\substack{n_1 > \cdots > n_d > 0 \\ n_j:\text{odd} \ \forall j}} \frac{2^d}{n_1^{k_1} \cdots n_d^{k_d}}$$

$$= \sum_{n_1 > \cdots > n_d > 0} \frac{(1-(-1)^{n_1}) \cdots (1-(-1)^{n_d})}{n_1^{k_1} \cdots n_d^{k_d}}$$

For example,

$$t(2) = \sum_{n>0} \frac{2}{(2n-1)^2}, \quad t(2,1) = \sum_{n>m>0} \frac{4}{(2n-1)^2(2m-1)}.$$

Multiple t-values (MtVs)

Remark.

MtVs satisfy the stuffle relations. For example,

$$t(2,1)t(3) = t(2,1,3) + t(2,3,1) + t(3,2,1) + 2t(5,1) + 2t(2,4).$$

Conjecture. Hoffman 2019

Let $\mathbf{MtV}_1 = \mathbb{Q}$ and $\mathbf{MtV}_w$ be the $\mathbb{Q}$-span of all MtVs of weight $w \geq 2$. Then

$$\sum_{w\geq 0} (\dim_{\mathbb{Q}} \mathbf{MtV}_{w+1})\, x^w = \frac{1}{1-x-x^2}.$$

Conjecture. Saha 2019

For all $w \in \mathbb{N}$, $\mathbf{MtV}_w$ has the basis

Multiple t-values (MtVs)

Theorem. Murakami 2021

If $s_1, \ldots, s_r \geq 2$ then $t(s_1, \ldots, s_r) \in \mathbf{MZV}$. Moreover, $\big\{t(s_1, \ldots, s_r) : s_1, \ldots, s_r \in \{2, 3\}\big\}$ generates $\mathbf{MZV}$ over $\mathbb{Q}$.

Theorem. Charlton 2021

On the motivic level Saha's elements are $\mathbb{Q}$-linearly independent, and that the (suitably regularized) elements $t^{\mathfrak{m}}(k_1, \ldots, k_d)$ $(k_1, \ldots, k_d \in \{1, 2\})$ form a basis for both the (extended) motivic MtVs and the Euler sums.

Multiple T-values (MTVs), Kaneko and Tsumura 2020

Definition.

For all admissible $(k_1, \ldots, k_d) \in \mathbb{N}^d$, the multiple T-values (MTV) are defined by

$$T(k_1, \ldots, k_d) := \sum_{\substack{n_1 > \cdots > n_d > 0 \\ n_i \equiv d-i+1 \bmod 2}} \frac{2^d}{n_1^{k_1} \cdots n_d^{k_d}}.$$

For example,

$$T(2) = t(2) = \sum_{n>0} \frac{2}{(2n-1)^2}, \quad T(2,1) = \sum_{n>m>0} \frac{4}{(2n)^2(2m-1)}.$$

Theorem. Kaneko and Tsumura 2020

MTVs satisfy the shuffle relations and the duality relation.

Multiple T-values (MTVs)

Theorem. Kaneko and Tsumura 2018

For all $w \in \mathbb{Z}_{\geq 3}$,

$$\sum_{j=2}^{w-1} 2^{j-1} T(j, w-j) = (w-1)T(w).$$

Conjecture.

For all $w \in \mathbb{Z}_{\geq 4}$,

$$\sum_{a+b+c=w} 2^b(3^{a-1}-1)T(a,b,c) = \frac{2}{3}(w-1)(w-2)T(w).$$

Theorem. Berger, Chandra, Jain, Xu, Xu and Z. 2022

The above conjecture is true.

Multiple T-values (MTVs)

Conjecture. Kaneko and Tsumura 2018

(1) For even weights, other than the single T-value $T(k)$, only $T(p, q, r)$ with p, r: odd ≥ 3 and q: even (and their duals) are in **MZV**.
(2) If the weight is odd, other than the single and the double T-values, only $T(p, 1, r)$ with p, r : even (and their duals) are in **MZV**.

Theorem. Murakami 2021

The conditions in Kaneko–Tsumura Conjecture above are sufficient.

Theorem.

For depth ≤ 3 **motivic** T-values the conditions in Kaneko–Tsumura Conjecture above are necessary.

Euler sums of depth two

Proposition.

Let $m, n \in \mathbb{N}$ and $\eta_1, \eta_2 = \pm 1$. Then the double zeta value $\zeta(m, n; \eta_1, \eta_2)$ of weight $m + n = w = 2K + 1$ is given by

$$\begin{aligned}
&\zeta^{\mathfrak{m}}(m, n; \eta_1, \eta_2) \\
&= (-1)^m \sum_{s=0}^{K-1} \left[\binom{w-2s-1}{m-1} \zeta^{\mathfrak{m}}(w-2s; \eta_1) \zeta^{\mathfrak{m}}(2s; \eta_1\eta_2) \right. \\
&\left. + \binom{w-2s-1}{n-1} \zeta^{\mathfrak{m}}(w-2s; \eta_2) \zeta^{\mathfrak{m}}(2s; \eta_1\eta_2) \right] \\
&+ \delta_{2|n} \zeta^{\mathfrak{m}}(m; \eta_1) \zeta^{\mathfrak{m}}(n; \eta_2) - \frac{1}{2} \zeta^{\mathfrak{m}}(w; \eta_1\eta_2).
\end{aligned}$$

Euler sums of depth two

Corollary.

For odd $w = m + n$ we have

$$t^{\mathfrak{l}}(m,n) = t^{\mathfrak{l}}(w),$$
$$T^{\mathfrak{l}}(m,n) = (-1)^n \binom{w-1}{m-1} T^{\mathfrak{l}}(w),$$
$$S^{\mathfrak{l}}(m,n) = (-1)^n \binom{w-1}{n-1} S^{\mathfrak{l}}(w).$$

Definition.

For any admissible composition $\boldsymbol{k} = (k_1, k_2, \dots, k_d)$, the multiple S-values (MSVs) is defined by

$$S(k_1, \dots, k_d) := M(\dots, \check{k}_{d-3}, k_{d-2}, \check{k}_{d-1}, k_d).$$

Multiple mixed values (MMVs), Xu and Z. 2022

Motivation.

Can we find a common generalization of MtVs, MTVs and MSVs so that the DBSF holds?

Definition.

Let $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_d) \in \{\pm 1\}^d$ and $\boldsymbol{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$ be admissible. Define the multiple mixed values (MMVs) by

$$M(\boldsymbol{k}; \boldsymbol{\epsilon}) := \sum_{m_1 > \cdots > m_d > 0} \frac{(1 + \epsilon_1(-1)^{m_1}) \cdots (1 + \epsilon_d(-1)^{m_d})}{m_1^{k_1} \cdots m_d^{k_d}}$$

$$= \sum_{\substack{n_1 > \cdots > n_d > 0 \\ 2 | n_j \text{ if } \epsilon_j = 1 \\ 2 \nmid n_j \text{ if } \epsilon_j = -1}} \frac{2^d}{n_1^{k_1} \cdots n_d^{k_d}}.$$

$(\epsilon_1, \dots, \epsilon_d)$: parity signatures.

MMVs

Convention

We put a check on top of s_j if $\epsilon_j = -1$.

$$M(\check{2}) = M(2; -1) = \sum_{m \geq 1} \frac{2}{(2m-1)^2} = t(2) = T(2),$$

$$M(3, \check{1}) = M(3, 1; 1, -1) = \sum_{n>m>0} \frac{4}{(2n)^3(2m-1)} = T(3, 1),$$

$$M(\check{6}, 2) = M(3, 1; -1, 1) = \sum_{n>m>0} \frac{4}{(2n-1)^6(2m)^2} = S(6, 2).$$

Main problems.

Find all the $\mathbb{Q}$-linear relations among MMVs.

1. Duality
2. DBSF

Algebraic setup.

- For $k \in \mathbb{N}$ and $\epsilon = \pm 1$, put $\mathsf{z}_{k,\epsilon} := \omega_0^{k-1}\omega_\epsilon$, where

$$w_0 := \frac{dt}{t}, \quad w_{-1} := \frac{2dt}{1-t^2}, \quad w_1 := \frac{2tdt}{1-t^2}.$$

- $\mathsf{X} := \{\omega_0, \omega_1, \omega_{-1}\}$ alphabet
- X^*: words over X including the empty word $\mathbf{1}$
- $|\mathbf{w}|$: *weight* of $\mathbf{w} \in \mathsf{X}^*$
- $\mathrm{dep}(\mathbf{w})$: *depth*, i.e., number of $\omega_{\pm 1}$'s contained in $\mathbf{w}$
- $\mathfrak{A}$: (weight) graded noncommutative polynomial $\mathbb{Q}$-algebra generated by X^*
- $\mathfrak{A}^0$: subalgebra of $\mathfrak{A}$ generated by *admissible words*, i.e., those beginning with ω_0 and ending with $\omega_{\pm 1}$.
- $\mathfrak{A}_ш$: $\mathfrak{A}$ equipped with the multiplication ш

DBSF Relations of MMVs

Definition.

For an admissible word $\mathbf{w} = z_{k_1,\epsilon_1} \cdots z_{k_r,\epsilon_r} \in \mathfrak{A}^0$, we set

$$\mu(\mathbf{w}) := \int_0^1 \mathbf{w}, \qquad M(\mathbf{w}) := M(\boldsymbol{k}; \epsilon_1, \dots, \epsilon_r),$$

and $M(\mathbf{1}) = \mu(\mathbf{1}) = 1$. We then extend μ to $\mathfrak{A}^0$ by $\mathbb{Q}$-linearity.

Proposition.

The map $\mu : \mathfrak{A}^0_{\sqcup\!\sqcup} \longrightarrow \mathbb{R}$ is an algebra homomorphism.

Remark.

As for MZV and ES we can define a stuffle structure $\mathfrak{A}_*$ and then the DBSF.

Theorem (Duality Relation). Xu and Z. 2022

Let $\boldsymbol{k} = (k_1, \dots, k_d) \in (\mathbb{Z}_{\geq 0})^d, \boldsymbol{l} = (l_1, \dots, l_d) \in \mathbb{N}^d$ and $\boldsymbol{\epsilon} \in \{\pm 1\}^d$. Then for all admissible values

$$M\big(\omega_0^{k_1}\omega_{\epsilon_1}^{l_1}\omega_0^{k_2}\omega_{\epsilon_2}^{l_2}\cdots\omega_0^{k_d}\omega_{\epsilon_d}^{l_d}\big) = M\big(u_{\epsilon_d}^{l_d}\omega_{-1}^{k_d}\cdots u_{\epsilon_2}^{l_2}\omega_{-1}^{k_2}u_{\epsilon_1}^{l_1}\omega_{-1}^{k_1}\big),$$

where $u_{-1} = \omega_0$ and $u_1 = \omega_0 + \omega_1 - \omega_{-1}$.

Proof. Use the substitution $t \to \frac{1-t}{1+t}$. Then

$$\omega_0 = d\log t \to d\log\frac{1-t}{1+t} = -\omega_{-1},$$
$$\omega_{-1} \to -\omega_0,$$
$$\omega_1 \to -\big(\omega_0 + \omega_1 - \omega_{-1}\big).$$

Duality Relations of MMVs

Example

In weight 4, we have the duality relation

$$M(2,1,\check{1}) = \int_0^1 \omega_0\omega_1\omega_{-1}^2 = \int_0^1 \omega_0^2(\omega_0+\omega_1-\omega_{-1})\omega_{-1}$$
$$= M(\check{4}) + M(\check{3},\check{1}) - M(3,\check{1}).$$

Remark.

It is possible to consider *regularized* duality by allowing ω_1 at the beginning.

w	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathbf{MtV}_w$	0	**1**	1	2	3	5	8	13	21	34	55	89	144
$\dim \mathbf{MTV}_w$	**1**	0	1	1	2	2	4	5	9	10	19	23	42
$\dim \mathbf{MSV}_w$	**1**	0	1	2	3	4	6	10	15	22	32	52	76
$\dim \mathbf{MMV}_w$	0	0	1	2	4	7	12	20	33	54	88	143	232
$\dim \mathbf{MMVe}_w$	0	0	1	2	4	7	12	20	33	54	88	143	232
$\dim \mathbf{MMVo}_w$	0	0	1	2	4	6	10	16	27	44	73	120	198

Theorem. (Xu and Z. 2022)

Let $k \in \mathbb{N}$, $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$. Then

$$\dim \mathbf{MMVe}_k \le \dim \mathbf{MMV}_k \le F_k - 1.$$

Remark.

The codimension one subspace of $\mathbf{MMV}_k$ in the Euler sum space $\mathbf{ES}_k$ should be generated by $\log^k 2$.

Dimensions

w	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathbf{MtV}_w$	0	**1**	1	2	3	5	8	13	21	34	55	89	144
$\dim \mathbf{MTV}_w$	**1**	0	1	1	2	2	4	5	9	10	19	23	42
$\dim \mathbf{MSV}_w$	**1**	0	1	2	3	4	6	10	15	22	32	52	76
$\dim \mathbf{MMV}_w$	0	0	1	2	4	7	12	20	33	54	88	143	232
$\dim \mathbf{MMVe}_w$	0	0	1	2	4	7	12	20	33	54	88	143	232
$\dim \mathbf{MMVo}_w$	0	0	1	2	4	6	10	16	27	44	73	120	198

Conjecture.

Let $k \in \mathbb{N}$, $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$. Then

$$\dim \mathbf{MMV}_k = \dim \mathbf{MMVe}_k = F_k - 1,$$

$$\dim \mathbf{MTV}_{2k} = \dim_{\mathbb{Q}} \mathbf{MTV}_{2k-1} + \dim_{\mathbb{Q}} \mathbf{MTV}_{2k-2}$$

(Kaneko-Tsumura),

$$\dim \mathbf{MSV}_{2k+1} = \dim_{\mathbb{Q}} \mathbf{MSV}_{2k-1} + 2\dim_{\mathbb{Q}} \mathbf{MSV}_{2k-2}$$

(M. Kobayashi).

Alternating MMVs, Xu, Yan and Z. 2023

Definition.

For any $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in \{\pm 1\}^r$, and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r) \in \{\pm 1\}^r$ with $(k_1, \sigma_1) \neq (1, 1)$ we define the alternating multiple mixed values (AMMVs) by

$$M_{\boldsymbol{\sigma}}^{\epsilon}(\boldsymbol{k}) := \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{(1 + \epsilon_j (-1)^{m_j}) \sigma_j^{(2m_j + 1 - \epsilon_j)/4}}{m_j^{k_j}}.$$

$(\epsilon_1, \dots, \epsilon_d)$: parity signatures,
$\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_r)$: alternating signatures.

Example.

$$M(\check{\bar{a}}, \bar{b}, \check{c}, d) = M^{\text{od},\text{ev},\text{od},\text{ev}}_{-1,-1,1,1}(a, b, c, d)$$
$$= \sum_{\substack{m_1>m_2>m_3>m_4>0 \\ m_1,m_3\in\text{od},\ m_2,m_4\in\text{ev}}} \frac{16(-1)^{(m_1+1)/2}(-1)^{m_2/2}}{m_1^a m_2^b m_3^c m_4^d}.$$

Alternating MMVs

Theorem. Xu, Yan and Z. 2023

Set

$$w_0 := \frac{dt}{t}, \quad w_{+1}^{-1} := \frac{2dt}{1-t^2}, \quad w_{-1}^{-1} := \frac{-2dt}{1+t^2},$$
$$w_{+1}^{+1} := \frac{2tdt}{1-t^2}, \quad w_{-1}^{+1} := \frac{-2tdt}{1+t^2}.$$

Then for all $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ with $(k_1, \sigma_1) \neq (1,1)$, we have

$$M_{\boldsymbol{\sigma}}^{\epsilon}(\boldsymbol{k}) = \int_0^1 w_0^{k_1-1} w_{\sigma_1}^{\epsilon_1,\epsilon_2} w_0^{k_2-1} w_{\sigma_1\sigma_2}^{\epsilon_2,\epsilon_3} \cdots w_0^{k_r-1} w_{\sigma_1\sigma_2\cdots\sigma_r}^{\epsilon_r}$$

and

$$\int_0^1 w_0^{k_1-1} w_{\sigma_1}^{\epsilon_1} \cdots w_0^{k_r-1} w_{\sigma_r}^{\epsilon_r} = \pm M_{\sigma_1,\sigma_2\sigma_1,\ldots,\sigma_r\sigma_{r-1}}^{\epsilon_1\cdots\epsilon_r,\epsilon_2\cdots\epsilon_r,\ldots,\epsilon_{r-1}\epsilon_r,\epsilon_r}(\boldsymbol{k}).$$

Alternating MMVs

Example.

$$M(\check{3},\bar{2}) = M^{\mathrm{od,ev}}_{1,-1}(3,2) = \sum_{n_1>n_2>0} \frac{4(-1)^{n_2}}{(2n_1-1)^3(2n_2)^2}$$
$$= \int_0^1 w_0^2 w_{+1}^{-1} w_0 w_{-1}^{+1},$$
$$M(\bar{2},3,\check{\bar{4}}) = \sum_{n_1>n_2>n_3>0} \frac{8(-1)^{n_1+n_3-1}}{(2n_1-2)^2(2n_2-2)^3(2n_3-1)^4}$$
$$= \int_0^1 w_0 w_{-1}^{+1} w_0^2 (-w_{-1}^{-1}) w_0^3 w_{+1}^{-1}.$$

Theorem (Regularized DBSF).

The AMMVs satisfy finite DBSF. These relations can be extended to regularized DBSF.

Alternating MMVs

Theorem (Duality Relations).

Let $\boldsymbol{k} = (k_1, \dots, k_r), \boldsymbol{l} = (l_1, \dots, l_r) \in \mathbb{N}^r$ and $\boldsymbol{\sigma}, \boldsymbol{\epsilon} \in \{\pm 1\}^r$. Then for admissible values

$$M\Big(\omega_0^{k_r}(\omega_{\sigma_r}^{\epsilon_r})^{l_r} \cdots \omega_0^{k_1}(\omega_{\sigma_1}^{\epsilon_1})^{l_1}\Big) = M\Big((u_{\sigma_1}^{\epsilon_1})^{l_1} u_0^{k_1} \cdots (u_{\sigma_r}^{\epsilon_r})^{l_r} u_0^{k_r}\Big),$$

where $u_0 = w_{+1}^{-1}$, $u_{+1}^{-1} = w_0$, $u_{-1}^{-1} = w_{-1}^{-1}$, $u_{+1}^{+1} = w_0 + w_{+1}^{+1} - w_{+1}^{-1}$ and $u_{-1}^{+1} = w_{+1}^{+1} - w_{+1}^{-1} - w_{-1}^{+1}$.

Example.

$$\begin{aligned} M(\check{\bar{2}}, \check{1}, \check{\bar{1}}) &= \int_0^1 w_0 w_{-1}^{+1} w_{-1}^{+1} w_{+1}^{-1} = \int_0^1 u_{+1}^{-1} u_{-1}^{+1} u_{-1}^{+1} u_0 \\ &= M(\check{2}, \check{1}, \check{1}) + M(\check{2}, 1, \check{1}) + M(2, \check{\bar{1}}, \check{\bar{1}}) \\ &+ M(\bar{2}, \bar{1}, \check{1}) + M(\check{\bar{2}}, \check{1}, \check{\bar{1}}) - M(2, 1, \check{1}) \\ &- M(\check{2}, \check{\bar{1}}, \check{\bar{1}}) - M(2, \check{1}, \check{1}) - M(\check{\bar{2}}, \check{\bar{1}}, \check{1}). \end{aligned}$$

Some application and evaluations of AMMVs

Example.

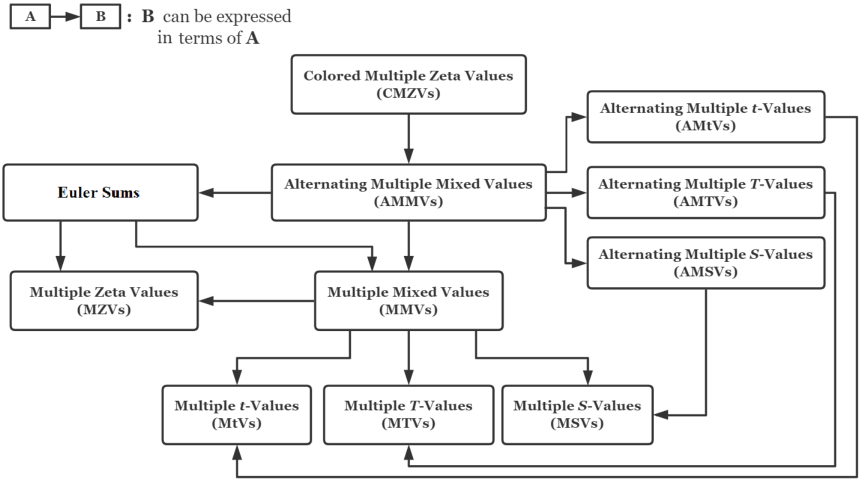
For positive integer r,

$$\int_0^1 \frac{\arctan^r(x)}{x}\,dx = (-1)^{[(r+1)/2]}\frac{r!}{2^r}T(\bar{2},\{1\}_{r-1}).$$

Example.

$$\begin{aligned}
T(\bar{3},1,\bar{1}) &= -\frac{1}{8}\pi^3 G - \frac{7}{32}\pi^2\zeta(3) + \frac{93}{16}\zeta(5),\\
S(\bar{3},1,\bar{1}) &= 2\mathrm{Li}_5(1/2) - \frac{589}{256}\zeta(5) - \frac{7}{8}\zeta(3)\log^2(2) - \frac{1}{60}\log^5(2)\\
&\quad + \frac{1}{36}\pi^2\log^3(2) + \frac{151}{5760}\pi^4\log(2),\\
S(4,1,\bar{1}) &= \zeta(\bar{5},1) + \frac{1}{12}\pi^2\mathrm{Li}_4(1/2) - \frac{83}{128}\zeta^2(3) + \frac{7}{32}\pi^2\zeta(3)\log(2)\\
&\quad - \frac{31}{\ }\zeta(5)\log(2) + \frac{227\pi^6}{\ } + \frac{1}{\ }\pi^2\log^4(2) - \frac{1}{\ }\pi^4\log^2(2)
\end{aligned}$$

Relations among different vector spaces



Dimensions of subspaces of **AMMV**

w	0	1	2	3	4	5	6
$\dim_{\mathbb{Q}} \mathbf{ES}_w$	1	1	2	3	5	8	13
$\dim_{\mathbb{Q}} \mathbf{AMtV}_w$	1	1	3	6	12	24	48
$\dim_{\mathbb{Q}} \mathbf{AMTV}_w$	1	1	2	4	7	13	24
$\dim_{\mathbb{Q}} \mathbf{AMSV}_w$	1	1	3	6	12	22	42
$\dim_{\mathbb{Q}} \mathbf{AMMV}_w$	1	2	4	8	16	32	64
$\dim_{\mathbb{Q}} \mathbf{CMZV}^4_w$	1	2	4	8	16	32	64

Table: Conjectural Dimensions of Various Subspaces of **AMMV**.

Theorem.

For any $w \in \mathbb{N}_0$, we have

$$\mathbf{AMMV}_w \otimes_{\mathbb{Q}} \mathbb{Q}[i] = \mathbf{CMZV}^4_w \otimes_{\mathbb{Q}} \mathbb{Q}[i].$$

Dimensions of subspaces of **AMMV**

w	0	1	2	3	4	5	6	7	8
$\dim_{\mathbb{Q}} \mathbf{AMZV}_w$	1	1	2	3	5	8	13	21	34
$\dim_{\mathbb{Q}} \mathbf{AMtV}_w$	1	1	3	6	12	24	48	96	192
$\dim_{\mathbb{Q}} \mathbf{AMTV}_w$	1	1	2	4	7	13	24	44	81
$\dim_{\mathbb{Q}} \mathbf{AMSV}_w$	1	1	3	6	12	22	42	80	156

Table: Conjectural Dimensions of Various Subspaces of **AMMV**.

Conjecture.

Set $\mathbf{AMtV}_0 = \mathbb{Q}$. We have the following generating function

$$\sum_{n=0}^{\infty} (\dim_{\mathbb{Q}} \mathbf{AMtV}_n) t^n = \frac{1 - t + t^2}{1 - 2t}.$$

Dimensions of subspaces of **AMMV**

w	0	1	2	3	4	5	6	7	8
$\dim_{\mathbb{Q}} \mathbf{AMSV}_w$	1	1	3	6	12	22	42	80	156

Table: Conjectural Dimensions of Various Subspaces of **AMSV**.

Conjecture.

We have $\dim_{\mathbb{Q}} \mathbf{AMSV}_1 = 1$, $\dim_{\mathbb{Q}} \mathbf{AMSV}_2 = 3$, and

$$\dim_{\mathbb{Q}} \mathbf{AMSV}_n = 2 \dim_{\mathbb{Q}} \mathbf{AMSV}_{n-1} - 2 \left\lfloor \frac{n-3}{2} \right\rfloor \quad \text{for all } n \geq 3.$$

Dimensions of subspaces of **AMMV**

w	0	1	2	3	4	5	6	7	8
$\dim_{\mathbb{Q}} \mathbf{AMTV}_w$	1	1	2	4	7	13	24	44	81

Table: Conjectural Dimensions of **AMTV**.

Conjecture.

Set $\mathbf{AMTV}_0 = \mathbb{Q}$. We have the following generating function

$$\sum_{n=0}^{\infty} (\dim_{\mathbb{Q}} \mathbf{AMTV}_n) t^n = \frac{1}{1 - t - t^2 - t^3}.$$

Namely, the dimensions form the tribonacci sequence $\{d_w\}_{w \geq 1} = \{1, 2, 4, 7, 13, 24, \dots\}$, see A000073 at oeis.org.

Finite AMMV

Definition.

Let $\mathcal{P}$ be the set of prime numbers. Define

$$\mathcal{A} = \Big(\prod_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}\Big) \Big/ \Big(\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p\mathbb{Z}\Big)$$

with componentwise addition and multiplication.

Lemma.

$\mathbb{Q}$ can be embedded into $\mathcal{A}$ as a sub-algebra.

Remark.

We can define algebraic and transcendental numbers in $\mathcal{A}$ over $\mathbb{Q}$.

Finite AMMVs

Definition.

For any $\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_r) \in \{\pm 1\}^r$, and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r) \in \{\pm 1\}^r$ we define the finite AMMVs by

$$M_{\mathcal{A}}(\boldsymbol{k}; \boldsymbol{\epsilon}; \boldsymbol{\sigma}) = \big(M_p(\boldsymbol{k}; \boldsymbol{\epsilon}; \boldsymbol{\sigma})\big)_{p \in \mathcal{P}} \in \mathcal{A}.$$

Example

When all $\boldsymbol{\epsilon} = (1, 1, \ldots, 1)$ we get finite MZVs. When all $\boldsymbol{\epsilon} = \boldsymbol{\sigma} = (1, 1, \ldots, 1)$ we get finite MZVs.

Conjecture. Kaneko & Zagier, 2014

There is an isomorphism

$$f_{KZ} : \mathbf{FMZV}_w \xrightarrow{\sim} \frac{\mathbf{MZV}_w}{\zeta(2)\mathbf{MZV}_{w-2}}$$

$$\zeta_{\mathcal{A}}(\boldsymbol{s}) \longmapsto \zeta_{ш}^{\mathcal{S}}(\boldsymbol{s}) \quad (\text{or } \zeta_{*}^{\mathcal{S}}(\boldsymbol{s}))$$

where

$$\zeta_{ш}^{\mathcal{S}}(s_1, \ldots, s_d) = \sum_{k=0}^{d} (-1)^{s_1+\cdots+s_k} \zeta_{ш}^{T}(s_k, \ldots, s_1) \zeta_{ш}^{T}(s_{k+1}, \ldots, s_d).$$

Definition-Lemma.

We call $\zeta_{ш}^{\mathcal{S}}(\boldsymbol{s})$ a ш-symmetrized MZV. It's a constant independent of T. Moreover, $\zeta_{ш}^{\mathcal{S}}(\boldsymbol{s}) \equiv \zeta_{*}^{\mathcal{S}}(\boldsymbol{s}) \pmod{\zeta(2)}$.

Theorem. Yasuda, 2014

The map f_{KZ} is surjective.

Finite MZVs

Conjecture. (Z., 2014)

There is an isomorphism

$$\mathbf{FES}_w \xrightarrow{\sim} \frac{\mathbf{ES}_w}{\zeta(2)\mathbf{ES}_{w-2}}$$

$$\zeta_{\mathcal{A}}\binom{\boldsymbol{s}}{\boldsymbol{\eta}} \longmapsto \zeta^{\mathcal{S}}_{\sqcup\!\sqcup}\binom{\boldsymbol{s}}{\boldsymbol{\eta}}$$

where

$$\zeta^{\mathcal{S}}_{\sqcup\!\sqcup}\binom{\boldsymbol{s}}{\boldsymbol{\eta}} := \sum_{k=0}^{d}\left(\prod_{j=1}^{k}(-1)^{s_j}\eta_j\right)\zeta^{T}_{\sqcup\!\sqcup}\binom{s_k,\dots,s_1}{\eta_k,\dots,\eta_1}\zeta^{T}_{\sqcup\!\sqcup}\binom{s_{k+1},\dots,s_d}{\eta_{k+1},\dots,\eta_d}.$$

Example.

$$\zeta_{\mathcal{A}}(\bar{1}) = -2q_2 \longmapsto \zeta^{\mathcal{S}}_{\sqcup\!\sqcup}(\bar{1}) = 2\zeta(\bar{1}) = -2\log 2.$$

Summary

Main Results

- Defined a level four variant of AMZVs (i.e., Euler sums) containing both AMtVs and AMTVs, called MMVs.
- Found the regularized DBSF and duality relations of AMMVs.
- Studied some subspaces of AMMVs and conjectured their dimensions.
- Began to investigate the finite AMMVs