ALGEBRA II Problems: Week 11

Epiphany Term 2012

Homework for Tuesday, Jan. 24: **Q1, 5, 6**.
For tutorials: problems 2, 3, 4, 7, 8.

1. (i) Show that if x and y are elements of finite order of a group G, and $xy = yx$, then xy is also an element of finite order. What can you say about the order of xy in terms of the orders of x and y?
   (ii) Show that the elements of finite order in an abelian group form a subgroup.
   (iii) Find a group G and elements x, y of G such that x and y have finite order yet xy has infinite order.
2. Which of the following functions are (i) injective (ii) surjective (iii) bijective?
   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 + x$.
   (b) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = 2x$.
   (c) $f : [0, 2] \to [0, 1],\ f(x) = \sin x$.
   (d) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x(x + 1)/2$.
3. Which of the following functions are homomorphisms from the multiplicative group of non-zero real numbers to itself?
   (a) $x \mapsto |x|$; (b) $x \mapsto -x$; $x \mapsto 2x$; (d) $x \mapsto x^2$; (e) $x \mapsto -1/x$.
4. Decompose D_6 into left cosets with respect to the subgroup $\{e, r^3, s, sr^3\}$. Is every left coset also a right coset?
5. Find the centre of the quaternion group Q_8, which is given as a set (denoting the identity e by 1) by $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with the relations $ij = k = -ji$ and $i^2 = j^2 = k^2 = -1$ (as well as $(-1)^2 = 1$, $-1 \neq 1$, as usual).
6. Show that every subgroup of the quaternion group Q_8 (see Problem 5) is normal.
7. Let A be an abelian group and let $D = \{(a, a) \mid a \in A\}$.
   Show that
   $$(A \times A)/D \cong A \qquad (*).$$
   [Hint: find a suitable group homomorphism and invoke the First Isomorphism Theorem for groups.]
   Show also that statement $(*)$ is not true in general for non-abelian A.
8. Let G and H be groups. Show that the set $K = \{ (g, e) \mid g \in G \}$ is a normal subgroup of $G \times H$, that K is isomorphic to G, and that $(G \times H)/K$ is isomorphic to H.