ALGEBRA II Problems: Week 11 (Group check)

Epiphany Term 2014

Hwk problems **1(2), 1(4), 1(5), 1(7), 1(11)** due: Tuesday, Jan. 28 during the lectures. For tutorials in week 11: the remaining items under 1.

1. For the following pairs $(S, \circ)$ with S a set and $\circ$ a binary operation on S (i.e. a map $\circ : S \times S \to S$), determine whether it defines a group or not. [Give a proof or indicate what prevents the pair from forming a group. Provide your arguments—note that it may sometimes be easier if you can prove it to be a subgroup of an object that you know to be a group.] Moreover, give the neutral and inverse elements in each case.

   (1) $(S, \circ) = \big(\{\frac{a}{2} \mid a \in \mathbb{Z}\}, +\big)$;
   (2) for S the rational numbers $b/2^a$ with b odd or $b = 0$, and $a \in \{1, 2, 3, \dots\}$, and $\circ = +$;
   (3) for S the rational numbers with denominators of the form 2^a, $a \in \{0, 1, 2, 3, \dots\}$, and $\circ = +$;
   (4) for S the rational numbers with denominators of the form $2^a 3^b$, $a, b \in \{0, 1, 2, 3, \dots\}$, and $\circ = +$;
   (5) for S the rational numbers whose denominator is *not* divisible by 2, and $\circ = +$;
   (6) for S the complex numbers of norm 1, i.e. the points on the unit circle in $\mathbb{C}$, and $\circ = \cdot$;
   (7) for S the complex numbers of norm 2^a, $a \in \mathbb{Z}$, and $\circ = \cdot$;
   (8) for S the complex numbers of norm 2^{2a+1}, $a \in \mathbb{Z}$, and $\circ = \cdot$;
   (9) for S the sequences of non-zero rational numbers $(r_1, r_2, \dots)$, i.e. with $r_i \in \mathbb{Q} \setminus \{0\}$, and where $\circ =$ "slotwise" multiplication, i.e. $(r_1, r_2, \dots) \circ (s_1, s_2, \dots) = (r_1 s_1, r_2 s_2, \dots)$;
   (10) for S the subset of vectors (a, b, c) in $\mathbb{R}^3$ for which $a = b$, and $\circ =$ vector addition;
   (11) for S the set of vectors in $\mathbb{R}^3$ and $\circ =$ cross product of two vectors;
   (12) for S the set of vectors in $\mathbb{R}^3$ and $\circ =$ scalar product of two vectors;
   (13) for the set $S = \{a, b\}$, with $\circ$ defined as follows
   $$a \circ b = b\,, \quad b \circ a = b\,, \quad a \circ a = a\,, \quad b \circ b = a\,;$$
   (14) for the set $S = \{a, b, c\}$, with $\circ$ defined as follows
   $$a \circ a = b \circ b = c \circ c = a\,,$$
   $$a \circ b = b \circ a = c\,, \quad b \circ c = c \circ b = a\,, \quad c \circ a = a \circ c = b\,;$$
   (15) for $S = \mathbb{Z}$ and $\circ$ defined for any $a, b \in \mathbb{Z}$ (using the usual addition on $\mathbb{Z}$) as follows:
   $$a \circ b = a + b + 1\,.$$

Challenge: For a fixed odd prime number p let S be the set of pairs $(a_1, a_2) \in \mathbb{Q} \times \mathbb{Q}$ where for $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ one defines $\circ$ as follows:

$$(a_1, a_2) \circ (b_1, b_2) = \Big(a_1 + b_1, a_2 + b_2 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a_1^i b_1^{p-i}\Big)\,.$$

Determine whether the pair $(S, \circ)$ forms a group. [Give a proof or counterexample.]