ALGEBRA II Problems: Week 12

Epiphany Term 2012

Hwk: Problems **1, 3, 10**, due Tuesday, Jan. 31, after the lecture.

1. Show that a k-cycle ($k \geq 2$) can be written as a product of transpositions:
$$(i_1\, i_2\, \ldots\, i_k) = (i_1\, i_k)(i_1\, i_{k-1}) \ldots (i_1\, i_2)\,.$$
For $k > 2$, find a different product of transpositions which does the job.

2. Express each of the following permutations as (i) a product of disjoint cycles and (ii) a product of transpositions:
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 6 & 4 & 1 & 8 & 2 & 3 & 5 \end{pmatrix}; \qquad (4568)(1245); \qquad (624)(253)(876)(45).$$

3. Let H be a subgroup of G. Show that gHg^{-1} is also a subgroup of G for any $g \in G$. Then show that every left coset of H is equal to a right coset of *some* subgroup (not necessarily H) of G.

4. How many different 5-cycles are there in S_5? [Justify your answer.]

5. Consider the following subset $W = \{\, e, (12)(34), (13)(24), (14)(23) \,\}$ of S_4.
   (a) Show that W forms a subgroup of S_4.
   (b) Is W isomorphic to $\mathbf{Z_4}$ or to $\mathbf{Z_2} \times \mathbf{Z_2}$? [Justify your answer.]
   (c) Show that W is isomorphic to the group of plane symmetries of a chess board.

6. Find the centre of S_n for $n \geq 3$.

7. Find the inverse of the cycle $(i_1\, i_2\, \ldots\, i_k)$.

8. Find a subgroup of S_4 which contains six elements. How many subgroups of order six are there in S_4? (Remember that a group of order six is isomorphic to either $\mathbf{Z_6}$ or S_3.)

9. For each of the groups $\mathbf{Z_6}$, S_3, D_4, $\mathbf{Z_2} \times \mathbf{Z_2}$ either find a subgroup of D_6 that is isomorphic to it, together with a specific isomorphism between the two, or explain why no such subgroup exists.

10. Let G be the set of all ordered triples (x, y, z) of real numbers. Show that the multiplication
$$(x, y, z)(x', y', z') = (x + x', y + y', xy' + z + z')$$
makes G into a group. Determine which of the following subsets of G are subgroups: $\{\, (x, y, z) \mid x = 0 \,\}$; $\{\, (x, y, z) \mid x = y \,\}$; $\{\, (x, y, z) \mid z = 0 \,\}$.
Is G abelian? Work out the nth power of (x, y, z) in G. Which elements of G have finite order?
Verify that $H = \{\, (x, y, z) \mid x = y = 0 \,\}$ is a subgroup of G and that $gh = hg$ for all $g \in G$, $h \in H$.