ALGEBRA II Problems: Week 12

Epiphany Term 2014

Homework for Thursday, Feb. 6: **Q1, 5, 6**.

1. (i) Show that if x and y are elements of finite order of a group G, and $xy = yx$, then xy is also an element of finite order. What can you say about the order of xy in terms of the orders of x and y?
   (ii) Show that the elements of finite order in an abelian group form a subgroup.
   (iii) Find a group G and elements x, y of G such that x and y have finite order yet xy has infinite order.
2. Which of the following functions are (i) injective (ii) surjective (iii) bijective?
   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3 + x$.
   (b) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = 2x$.
   (c) $f : [0, 2] \to [0, 1],\ f(x) = \sin x$.
   (d) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x(x + 1)/2$.
3. Which of the following functions are homomorphisms from the multiplicative group of non-zero real numbers to itself?
   (a) $x \mapsto |x|$; (b) $x \mapsto -x$; (c) $x \mapsto 2x$; (d) $x \mapsto x^2$; (e) $x \mapsto -1/x$.
4. Decompose D_6 into left cosets with respect to the subgroup $\{e, r^3, s, sr^3\}$. Is every left coset also a right coset?
5. The **centre** $Z(G)$ of a group G is the subset of elements $h \in G$ which commute with all elements in G, i.e. $Z(G) = \{h \in G \mid gh = hg \quad \forall g \in G\}$. Find the centre of the quaternion group Q_8, which is given as a set of 8 elements (denoting the identity e by 1) by $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with the relations $ij = k = -ji$ and $i^2 = j^2 = k^2 = -1$ (as well as $(-1)^2 = 1$, $-1 \neq 1$, as usual).
6. Show that every subgroup of the quaternion group Q_8 (see Problem 5) is normal.
7. Let G be the set of all ordered triples (x, y, z) of real numbers. Show that the multiplication

   $$(x, y, z)(x', y', z') = (x + x', y + y', xy' + z + z')$$

   makes G into a group. Determine which of the following subsets of G are subgroups: $\{ (x, y, z) \mid x = 0 \}$; $\{ (x, y, z) \mid x = y \}$; $\{ (x, y, z) \mid z = 0 \}$.

   Is G abelian? Work out the nth power of (x, y, z) in G. Which elements of G have finite order?

   Verify that $H = \{ (x, y, z) \mid x = y = 0 \}$ is a subgroup of G and that $gh = hg$ for all $g \in G$, $h \in H$.