ALGEBRA II Problems: Week 15

Epiphany Term 2012

Hwk: **Q1, 2** due Tuesday, Febr. 21, during the lectures.

1. The infinite dihedral group D_∞ is generated (as a subgroup of the group $S_{\mathbb{R}}$ of bijections : $\mathbb{R} \to \mathbb{R}$), by the translation $t(x) = x+1$ and the reflection $s(x) = -x$ of the real line. Work out its elements, and find the orbit and the stabilizer of each of the points 1, 1/2, 1/3.

2. Let H be a subgroup of a group G. Verify that the formula $(h, h')(x) = hxh'^{-1}$ defines an action of $H \times H$ on G. Find the orbit and the stabilizer of each element of G when $G = D_4$ and $H = \{ e, s \}$.

3. If G acts on X and H acts on Y prove that $G \times H$ acts on $X \times Y$ via

$$(g, h)((x, y)) = (g(x), h(y)).$$

Check that the orbit of (x, y) is $G(x) \times H(y)$ and that its stabilizer is $G_x \times H_y$. We call this action the *product action* of $G \times H$ on $X \times Y$.

4. Here are four group actions on $\mathbb{R}^4$.
(a) The usual action of $GL_4(\mathbb{R})$.
(b) Identify $\mathbb{R}^4$ with $\mathbb{R}^2 \times \mathbb{R}^2$ and take the product action of $SO_2 \times SO_2$.
(c) Think of $\mathbb{R}^4$ as $\mathbb{C} \times \mathbb{C}$ and let SU_2 act in the usual way.
(d) Identify $\mathbb{R}^4$ with $\mathbb{R}^3 \times \mathbb{R}$ and take the product action of $SO_3 \times \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathbb{R}$ by addition.
Discuss the structure of the orbits and the stabilizers in each case.

5. If G acts on X and on Y, show that the formula $g((x, y)) = (g(x), g(y))$ defines an action of G on $X \times Y$. Check that the stabilizer of (x, y) is the intersection of G_x and G_y. Give an example which shows this action need not be transitive even if G acts transitively on both X and Y. We call this action the *diagonal action* of G on $X \times Y$.
[A group action is said to be *transitive* if there is just one orbit.]

6. Let $X = \{1, 2, 3, 4\}$ and let G be the subgroup of S_4 generated by (1234) and (24). Work out the orbits and stabilizers for the diagonal action of G on $X \times X$.

7. Let $C = \{e^{ix} \mid 0 \le x < 2\pi\} \subset \mathbb{C}$ be the circle group, with composition inherited from the multiplication in $\mathbb{C}$. The group $C \times C$ is called the *torus*. Draw a picture of $C \times C$ to show why we give it this name. Describe the orbits of the following actions of $\mathbb{R}$ on the torus.
(a) The real number t sends (e^{ix}, e^{iy}) to $(e^{i(x+t)}, e^{iy})$.
(b) This time t sends (e^{ix}, e^{iy}) to $(e^{i(x+t)}, e^{i(y+t)})$.
(c) Finally agree that t sends (e^{ix}, e^{iy}) to $(e^{i(x+t)}, e^{i(y+t\sqrt{2})})$.

8. Calculate the number of elements conjugate to (12)(34)(56789) in S_{12}.