ALGEBRA II Problems: Week 16

Epiphany Term 2012

Hwk 1: **Q2,3** due Tuesday, Febr. 28 during the lectures.
Hwk 2: **Q10,11** due Tuesday, Mar 6 during the lectures.

1. Let n be *even*, and let D_n act on itself by conjugation. Find the orbits and stabilizers of the elements of D_n under this action.
In which sense does this case differ from the case when n is odd?

2. Let $V = \{e, (12)(34), (13)(24), (14)(23)\}$ act on A_4 (viewed as a subgroup of S_4, as usual) by conjugation.
Determine, for each $x \in A_4$, its orbit $V(x)$ and stabilizer V_x under this action and give a natural bijection between $V(x)$ and the set of cosets in V with respect to V_x.

3. Given an action of a group G on a set, show that every point of some orbit has the same stabilizer if and only if this stabilizer is a normal subgroup of G.

4. Show that the cosets of A_n in S_n are the set of even and the set of odd permutations. Deduce that if $n > 1$ then $|A_n| = \frac{1}{2}n!$.

5. Prove or give a counterexample to the following statement:
if a and b are two elements in a group G, then ab and ba have the same order.

6. Prove that if G is a finite group of *odd* order, then no $x \in G$, other than $x = e$, is conjugate to its inverse. Contrast this with the case of a dihedral group.

7.* Show that every group of order $4n + 2$ contains a subgroup of order $2n + 1$. [Hint: Use Cayley's Theorem and Cauchy's Theorem and think odd and even.]

8.* Let G be a finite group whose order is divisible by a prime p, and let p^m be the largest power of p which divides $|G|$. Let X denote the collection of all subsets of G which have p^m elements. Use the action of G on X in which $g \in G$ sends $A \in X$ to gA to show that G contains a subgroup of order p^m.
[Hints: First show that the size of X, and hence of some orbit $G(A)$ for some $A \in X$ under G, is not divisible by p, then consider the stabilizer G_A.]

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9. Calculate the number of elements conjugate to $(1\,2)(3\,4)(5\,6\,7\,8\,9)$ in S_{12}.

10. How many 5-cycles does the alternating group A_7 contain? Prove that these 5-cycles form a single conjugacy class in A_7. Work out the number of distinct conjugacy classes of 5-cycles in A_6. List a representative from each class and check that no two of these representatives are conjugate in A_6.

11. If n is odd show there are exactly two conjugacy classes of n-cycles in A_n each of which contains $(n-1)!/2$ elements. When n is even prove that the $(n-1)$-cycles in A_n make up two conjugacy classes each of which contains $(n-2)!n/2$ elements.

12. Find all normal subgroups of S_5.

13. Find all normal subgroups of A_5.

14. Show that O_3, the group of *orthogonal* (3×3)-matrices in $GL_3(\mathbb{R})$, is *not* a normal subgroup of $GL_3(\mathbb{R})$.