ALGEBRA II Problems: Week 17 (Conjugacy for S_n and A_n, Cauchy's Thm)

Epiphany Term 2014

Hwk: **Q1, 2** due Thursday, Mar 12 during the lectures.

1. Calculate the number of elements conjugate to $(1\,2)(3\,4)(5\,6\,7\,8\,9)$ in S_{12}.

2. Find all normal subgroups of S_5.

3. Find all normal subgroups of A_5.

4. How many 5-cycles does the alternating group A_7 contain? Prove that these 5-cycles form a single conjugacy class in A_7. Work out the number of distinct conjugacy classes of 5-cycles in A_6. List a representative from each class and check that no two of these representatives are conjugate in A_6.

5. If n is odd show there are exactly two conjugacy classes of n-cycles in A_n each of which contains $(n-1)!/2$ elements. When n is even prove that the $(n-1)$-cycles in A_n make up two conjugacy classes each of which contains $(n-2)!n/2$ elements.

6. Show that $O(3)$, the group of *orthogonal* (3×3)-matrices in $GL_3(\mathbb{R})$, is *not* a normal subgroup of $GL_3(\mathbb{R})$.

7. Let H be a subgroup of G and write X for the set of left cosets of H in G.
Show that the formula $g(xH) = gxH$ defines an action of G on X.
Prove that H is a normal subgroup of G if and only if every orbit of the induced action of H on X contains just one point.

8. Let G be a finite group and let p be the smallest prime which is a factor of $|G|$. Prove that a subgroup H whose *index* in G, defined as $\frac{|G|}{|H|}$, is equal to p must be a normal subgroup of G. (You may wish to try the previous Problem first.)

9. Let N be a normal subgroup of A_n, for $n \geq 3$.
Show that if N contains a 3-cycle then $N = A_n$.
[Hint: try to control the behavior of a given cycle $(a\,b\,c)$ under conjugation by a product of two transpositions involving a, b, c and possibly another number.]

10. Give all the groups of order 1382, up to isomorphism.

11. Give all the groups of order 289, up to isomorphism.

12.* Let G be a finite group whose order is divisible by a prime p, and let p^m be the largest power of p which divides $|G|$. Let X denote the collection of all subsets of G which have p^m elements. Use the action of G on X in which $g \in G$ sends $A \in X$ to gA to show that G contains a subgroup of order p^m.
[Hints: First show that the size of X, and hence of some orbit $G(A)$ for some $A \in X$ under G, is not divisible by p, then consider the stabilizer G_A.]