ALGEBRA II Problems: Week 18

Epiphany Term 2012

1. Let H be a subgroup of G and write X for the set of left cosets of H in G. Show that the formula $g(xH) = gxH$ defines an action of G on X.
Prove that H is a normal subgroup of G if and only if every orbit of the induced action of H on X contains just one point.

2. Let G be a finite group and let p be the smallest prime which is a factor of $|G|$. Prove that a subgroup H whose *index* in G, defined as $\frac{|G|}{|H|}$, is equal to p must be a normal subgroup of G. (You may wish to try the previous Problem first.)

3. Let N be a normal subgroup of A_n, for $n \geq 3$.
Show that if N contains a 3-cycle then $N = A_n$.
[Hint: try to control the behavior of a given cycle $(a\,b\,c)$ under conjugation by a product of two transpositions involving a, b, c and possibly another number.]

4. Give all the groups of order 1382, up to isomorphism.

5. Give all the groups of order 289, up to isomorphism.

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6. Write down the torsion coefficients of

$$(a)\ \mathbb{Z}_{15} \times \mathbb{Z}_2 \times \mathbb{Z}_{20}; \quad (b)\ \mathbb{Z}_{10} \times \mathbb{Z}_{36} \times \mathbb{Z}_{14} \times \mathbb{Z}_{21}.$$

7. Let G be an abelian group of order 100. Show that G must contain an element of order 10. What are the torsion coefficients of G if no element of G has order greater than 10?

8. Classify the abelian groups of order 32, 60 and 144.

9. If the order of a finite abelian group is not divisible by a square, show that the group must be cyclic.

10. Let G be a finite abelian group and write $A(q) = A_G(q)$ for the number of elements x of G which satisfy $x^q = e$. Find the torsion coefficients of G when $A(3) = 81$, $A(9) = 243$, $A(5) = 25$, $A(25) = 625$ and $x^{225} = e$ for all $x \in G$.

11. Find the rank and the torsion coefficients of the abelian group determined by generators $w,\ x,\ y,\ z$ and relations $3w + 5x - 3y = 0$, $\ 4w + 2x - 2z = 0$.

12. Find the rank and the torsion coefficients of the abelian group determined by generators $v,\ w,\ x,\ y,\ z$ and relations:

$$\begin{aligned} v - 7w + 14y - 21z &= 0; \\ 5v - 7w - 2x + 10y - 15z &= 0; \\ 3v - 3w - 2x + 6y - 9z &= 0; \\ v - w + 2y - 3z &= 0. \end{aligned}$$

13. How many elements of order (a) 3, (b) 9, (c) 4, (d) 12 does $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{36}$ contain?

14. Let G be a finite abelian group of order 360 which does not contain any elements of order 12 or 18. Find the torsion coefficients of G. How many elements of order 6 does G contain?