

Modular forms

Herbert Gangl

University of Durham

Lecture 2

Recall: we claimed that we can write $SL_2(\mathbb{Z})$ with *two* generators only:

Proposition

The group $SL_2(\mathbb{Z})$ is generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Also, we had

Theorem

A **fundamental domain** \mathcal{F} for the action of $SL_2(\mathbb{Z})$ is given by

$$\mathcal{F} = \left\{ \tau \in \mathbb{H}; |\tau| \geq 1, -\frac{1}{2} \leq \operatorname{Re}(\tau) \leq \frac{1}{2} \right\}.$$

In other words,

- 1 every point in \mathbb{H} is equivalent to a point in \mathcal{F} ;
- 2 points in the *interior* of \mathcal{F} are $SL_2(\mathbb{Z})$ -inequivalent.

By identifying the boundaries of the fundamental domain \mathcal{F} for $SL_2(\mathbb{Z})$ we saw

Proposition

As topological spaces, we have

$$X := SL_2(\mathbb{Z}) \backslash \mathbb{H} \simeq \mathbb{C}.$$

In fact, this statement is even true as (non-compact) complex curves/Riemann surfaces!

Definition

We write

$$\bar{\mathbb{H}} = \mathbb{H} \cup \{\infty\} \cup \mathbb{Q}.$$

One can extend the topology of \mathbb{H} to one of $\bar{\mathbb{H}}$, by introducing a local coordinate at infinity:

Local variable at ∞

The analytic structure at the cusp ∞ arises as follows. Define

$$\tau \mapsto q = e^{2\pi i\tau}$$

which maps $U_C \setminus \{\infty\}$ to the punctured disc of radius $e^{-2\pi C}$ around 0. Furthermore, map

$$\infty \mapsto 0.$$

Then q descends to a map on $\pi_{\mathrm{SL}_2(\mathbb{Z})}(U_C)$ and is bijective for sufficiently large C ($C \geq 1$). So

q is a local variable at ∞ for $\mathrm{SL}_2(\mathbb{Z})$.

Upshot (without proof)

This is crucial for making $\bar{X} = \mathrm{SL}_2(\mathbb{Z}) \backslash \bar{\mathbb{H}}$ into a compact Riemann surface.

SECTION 2: MODULAR FORMS FOR $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$

Definition 2.1

Let $k \in \mathbb{Z}$. A meromorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is called **weakly modular of weight k** for the full modular group Γ_1 if

$$f(\gamma\tau) = (c\tau + d)^k f(\tau) = j(\gamma, \tau)^k f(\tau) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1.$$

Remark 2.2

For the generators T and S this means

- $T : \quad f(\tau + 1) = f(\tau),$
- $S : \quad f(-\frac{1}{\tau}) = \tau^k f(\tau).$

In fact, by the cocycle relation(!), if f transforms as above for T and S , then it transforms accordingly for the whole group.

Remark 2.3

$$\gamma = \begin{pmatrix} -1 & \\ & -1 \end{pmatrix} : \quad f(\tau) = (-1)^k f(\tau). \quad \text{Thus: } k \text{ odd} \Rightarrow f = 0.$$

Definition 2.4

We call a weakly modular function a **meromorphic modular form of weight k** if it is in addition

meromorphic at ∞ .

How to determine meromorphicity?

While f is (for $k \neq 0$) not a function on $X = \Gamma_1 \backslash \mathbb{H}$, it is periodic (of period 1) and therefore defines a function in the local variable $q = e^{2\pi i\tau}$ in a punctured nbhd of ∞ . We then require that f is meromorphic in q , i.e.

$$f(\tau) = \sum_{n \gg -\infty} a_n q^n \quad \text{Im}(\tau) \gg 0, \text{ i.e., near } \infty.$$

Definition 2.5

We call $f(\tau) = \sum_n a_n q^n$ the **q -expansion of f** (at ∞) and the a_n its **Fourier coefficients**.

Definition 2.6

A meromorphic weakly modular form f is **holomorphic** at ∞ if

$$a_n = 0 \quad \text{for } n < 0.$$

In fact, in that latter case f is bounded as $v \rightarrow \infty$ and one has $f(\infty) = a_0$.

Lemma 2.7

Let f be weakly modular. Then

- 1 f is meromorphic (at ∞) if and only if f has at most exponential growth at ∞ : $f(u + iv) = O(e^{Cv})$ for some $C > 0$.
- 2 f is holomorphic (at ∞) if and only if f has subexponential (moderate) growth at ∞ : $f(u + iv) = O(e^{Cv})$ for all $C > 0$.

Definition 2.8

Let $k \in \mathbb{Z}$. A meromorphic modular form is called a **(holomorphic) modular form of weight k** for the full modular group Γ_1 if it is holomorphic on \mathbb{H} and also at ∞ .

So $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfies

- 1 f is holomorphic on \mathbb{H}
- 2 $f(\gamma\tau) = (c\tau + d)^k f(\tau)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1$
- 3 $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ (f is holomorphic at ∞ or f is of moderate growth)

Definition 2.9

A modular form is called a **cusp form** if it "vanishes at the cusp ∞ ", i.e., the constant Fourier coefficient vanishes:

$$a_0 = 0.$$

Alternatively, f is of rapid decay.

Notation

Write $M_k(\Gamma_1)$ for the vector space of modular forms of weight k for Γ_1 .

Write $S_k(\Gamma_1)$ for the vector space of cusp forms of weight k for Γ_1 .

Remark 2.10

Warning: A modular form f is not bounded as $v \rightarrow 0$. Instead, we have

$$f(\tau) = O(v^{-k}). \quad (v \rightarrow 0)$$

Indeed, this follows from $f(\tau) = \tau^{-k} f(-\frac{1}{\tau})$.

Lemma 2.11

Let $f \in M_k(\Gamma_1)$. Then

$$v^{k/2} |f(\tau)|$$

is Γ_1 -invariant.

Proof.

$$Im(\gamma\tau)^{k/2} |f(\gamma\tau)| = \left(\frac{Im(\tau)}{|c\tau+d|^2} \right)^{k/2} |c\tau+d|^k |f(\tau)| = Im(\tau)^{k/2} |f(\tau)|.$$

□

Proposition 2.12

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_1)$ be a cusp form of weight k . Then

- 1 $Im(\tau)^{k/2} |f(\tau)|$ is bounded on \mathbb{H} .
- 2 **Hecke's Bound:**

$$|a_n| \leq Cn^{k/2} \quad (\text{for some } C > 0)$$

Proof.

For (1), note that $Im(\tau)^{k/2}|f(\tau)|$ is bounded in the fundamental domain \mathcal{F} which is the union of a compact set (where f is clearly bounded) and a nbhd of ∞ where f is small by the condition on a cusp form. Hence $Im(\tau)^{k/2}|f(\tau)|$ is bounded on all of \mathbb{H} by the previous lemma.

So we have $|f(\tau)| \leq C'v^{-k/2}$ some $C' > 0$. For (2), we then see

$$|a_n| = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} f(u + iv)e^{-2\pi in\tau} du \right| \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(\tau)|e^{2\pi nv} du \leq C'v^{-k/2}e^{2\pi nv} \quad (\text{for all } v)$$

Now take $v = 1/n$. Hence $|a_n| \leq C'e^{2\pi}n^{k/2}$, as required. \square

Remark 2.13

The **Ramanujan-Petersson conjecture**, proved by Deligne as a consequence of his proof of the Weil conjecture (relationship to arithmetic algebraic geometry) states

$$|a_n| = O(n^{(k-1)/2}).$$