Modular forms

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Lecture 6

SECTION 6: MODULAR FORMS FOR CONGRUENCE SUBGROUPS

Throughout, N will denote a positive integer.

Definition 6.1 (Principal Congruence Subgroup)

We define the principal congruence subgroup of level N by

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z});\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\}.$$

In particular, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.

Lemma 6.2

We record some facts about $\Gamma(N)$: (exercise)

- $\Gamma(N)$ *is the kernel of the natural homomorphism* $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$*, in particular* $\Gamma(N)$ *is normal in* $\mathrm{SL}_2(\mathbb{Z})$*.*
- *This map is a surjection:* $\mathrm{SL}_2(\mathbb{Z})/\Gamma(N) \simeq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$*.*
- *The index of* $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)]$ *of* $\Gamma(N)$ *in* $\mathrm{SL}_2(\mathbb{Z})$ *is given by*

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right).$$

Definition 6.3 (Congruence Subgroup)

A congruence subgroup Γ of level N is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma(N)$.

Note: In particular, every congruence subgroup has finite index in $\mathrm{SL}_2(\mathbb{Z})$.
(Not all subgroups of $\mathrm{SL}_2(\mathbb{Z})$ of finite index are congruence subgroups, but this won't

Definition 6.4 (Important congruence subgroups aside from $\Gamma(N)$)

The Hecke subgroup:

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z});\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{N} \right\}.$$

and

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z});\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{N} \right\}.$$

So we have $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset SL_2(\mathbb{Z})$.

Lemma 6.5 (exercise)

- *$\Gamma(N)$ is normal in $\Gamma_1(N)$ with $\Gamma_1(N)/\Gamma(N) \simeq \mathbb{Z}/N\mathbb{Z}$.*
- *$\Gamma_1(N)$ is normal in $\Gamma_0(N)$ with $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^*$.*
- $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right)$.

Lemma 6.6 (Fundamental domain, exercise)

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be any subgroup (not necessarily of finite index). Let $\mathcal{F}$ be a (say, the standard) fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$.
Then

$$\bigcup_{\gamma \in \overline{\Gamma} \backslash \mathrm{PSL}_2(\mathbb{Z})} \gamma \mathcal{F}$$

is a fundamental domain for the action of Γ, where the $\gamma \in \overline{\Gamma} \backslash \mathrm{PSL}_2(\mathbb{Z})$ runs over a set of right coset representatives of $\mathrm{PSL}_2(\mathbb{Z})$ modulo Γ.

(In the textbooks one often uses left coset rep's instead ... and then uses the inverses ... which are right coset rep's.)

Example 6.7

The index of $\Gamma_0(2)$ in $\mathrm{SL}_2(\mathbb{Z})$ is 3 with right coset rep's given by

$$\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \qquad S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}, \qquad ST=\begin{pmatrix}0&-1\\1&1\end{pmatrix}$$

Then a fundamental domain for $\Gamma_0(2)$ looks as follows:

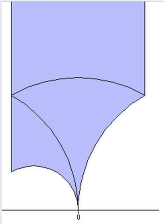

Figure: Fundamental domain for $\Gamma_0(2)$ (source: Carlo Angelantonj, Rankin-Selberg methods)

Note that $\Gamma_0(2)$ now has two (equivalence classes of) cusps.

Lemma 6.8 (Cusps of Γ)

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N. Then Γ acts on the set of (rational) cusps $\{\infty\}\cup\mathbb{Q}$ with finitely many orbits. (Pf: Γ has finite index in $\mathrm{SL}_2(\mathbb{Z})$.) Often we call the Γ-equivalence classes of (rational) cusps simply the cusps of Γ.

Example 6.9 (exercise)

1. $\Gamma_0(p)$ has two cusps, ∞ and 0.
2. $\Gamma_0(4)$ has 3 cusps: ∞, 0, $\frac{1}{2}$.

Lemma 6.10

$\Gamma_0(4)$ *is generated by* $\pm T = \pm \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ *and* $\pm ST^{-4}S = \pm \left(\begin{smallmatrix} 1 & \\ 4 & 1 \end{smallmatrix}\right)$*.*

Proof (omitted in lecture).

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$. We have

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ c & nc+d \end{smallmatrix}\right)$$

which shows that, unless $c = 0$, we can change the bottom row to $(c' d')$ with $|d'| < |c'|/2$.

Now

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \\ 4n & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} * & * \\ c+4nd & d \end{smallmatrix}\right)$$

which shows that unless $d = 0$, we can change the bottom row to (c', d') with $|c'| < 2|d'|$.

Each multiplication reduces the positive integer quantity $\min\{|c|, 2|d|\}$, so the process of alternating right multiplying with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & n \\ & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \\ 4n & 1 \end{smallmatrix}\right)$ must stop with $c' = 0$ or $d' = 0$.

If $c = 0$, then we have obtained $\pm \left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)$.

The case $d = 0$ can actually not happen ($nc + d$ is always odd). □

Definition 6.11 (Slash Operator)

For f a function on $\mathbb{H}$, we define the slash operator of weight k as

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau) \qquad (\gamma \in \mathrm{SL}_2(\mathbb{R})).$$

This defines an action of the group $\mathrm{SL}_2(\mathbb{R})$ on the space of functions on $\mathbb{H}$:

$$(f|_k\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2 \,.$$

So modularity of weight k for Γ means $f|_k\gamma = f$ for all $\gamma \in \Gamma$.

Definition 6.12

A function f on $\mathbb{H}$ is called weakly modular (of level N) for the congruence subgroup Γ (of level N) of weight k if

$$f|_k\gamma = f \qquad (\text{for all } \gamma \in \Gamma).$$

Definition 6.13 (Holomorphy/Meromorphy at ∞)

A weakly modular form f of level N is periodic with period dividing N (since $\left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N) \subseteq \Gamma$). Hence f is a function of $q_N = e^{2\pi i\tau/N}$, that is, on the punctured disc around 0. Then we call f meromorphic at ∞ if it has the Laurent expansion (Fourier expansion)

$$f(\tau) = \sum_{n \gg -\infty} a_n q_N^n \qquad (Im(\tau) \gg 0, \text{ i.e., near } \infty).$$

We call f holomorphic at ∞ if $a_n = 0$ for $n < 0$.

Definition 6.14 (Modular forms for Congruence subgroups)

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and let k be an integer. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k *with respect to* Γ if

(i) f is holomorphic on $\mathbb{H}$,

(ii) $f|_k\alpha = f$ for all $\alpha \in \Gamma$,

(iii) $f|_k\alpha$ is holomorphic at ∞ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$.

If, in addition, we have

- $a_0 = 0$ in the Fourier expansion of $f|_k\alpha$ for all $\alpha \in \mathrm{SL}_2(\mathbb{Z})$,

then we call f a cusp form of weight k with respect to Γ.

Notation

Write $M_k(\Gamma)$ for the vector space of modular forms of weight k for Γ.

Write $S_k(\Gamma)$ for the vector space of cusp forms of weight k for Γ.

If $\Gamma = \Gamma_0(N)$, we often write $M_k(N)$ and $S_k(N)$.

Remark 6.15

If $-I \notin \Gamma$, non-zero modular forms of odd weight may well exist, in contrast to $\mathrm{SL}_2(\mathbb{Z})$.

Remark 6.16

The condition on $f|_k\alpha$ at ∞ is also stated as “being holomorphic/vanishes at all cusps". The condition only depends on the Γ-equivalence classes of cusps and hence only needs to be checked for finitely many α.

Recall from the exercises that we have defined

$$\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \int_{\mathcal{F}_\Gamma} 1\,\frac{du dv}{v^2} =: \int_{\Gamma\backslash\mathbb{H}} 1\,\frac{du dv}{v^2},$$

where $\mathcal{F}_\Gamma$ is a (nice) fundamental domain of Γ in $\mathbb{H}$. We also write $d\mu(\tau)$ for the $\mathrm{SL}_2(\mathbb{R})$-invariant measure $\frac{dudv}{v^2}$ on $\mathbb{H}$: $d\mu(g\tau) = d\mu(\tau)$. Then

Lemma 6.17 (Exercise)

$$\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}) = \tfrac{\pi}{3}\,, \qquad \mathrm{vol}(\Gamma\backslash\mathbb{H}) = [\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]\,\tfrac{\pi}{3}\,.$$

Proposition 6.18

Let f be a non-zero meromorphic modular form of weight k for a congruence subgroup Γ. Then

$$\sum_{P\in\Gamma\backslash\overline{\mathbb{H}}} \frac{1}{N_P}\,ord(f) = \frac{k}{4\pi}\,\mathrm{vol}(\Gamma\backslash\mathbb{H}),$$

where N_P denotes the order of the stabilizer for the point $P\in\mathbb{H}$ in $\overline{\Gamma}$. For P a cusp we get $N_P = 1$, and the order of f at P is measured in terms of the local variable q_h (which could be half integral in case of an irregular cusp).

As in the $SL_2(\mathbb{Z})$ case we then obtain

Theorem 6.19

For Γ a congruence subgroup, we have

$$\dim M_k(\Gamma) \leq \frac{k}{4\pi} \operatorname{vol}(\Gamma\backslash\mathbb{H}) + 1.$$

Proof of Proposition 6.18.

We first assume that k is even. Then considering if necessary the group generated by Γ and ± 1, we can assume that $-1 \in \Gamma$ (this does not change either side of the asserted equation). Then $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}]$ and rep's for $\overline{\Gamma}\backslash\,\mathrm{PSL}_2(\mathbb{Z})$ (which give rise to a fundamental domain for Γ ...) lift to rep's for $\Gamma\backslash\,\mathrm{SL}_2(\mathbb{Z})$. Consider

$$F(\tau) := \prod_{\gamma \in \Gamma\backslash\,\mathrm{SL}_2(\mathbb{Z})} (f|_k\gamma)(\tau).$$

Then F is a non-zero meromorphic form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $[\mathrm{PSL}_2(\mathbb{Z}) : \overline{\Gamma}] \cdot k$ (exercise). We apply the $k/12$-formula for F (which is Prop. 6.18 for $\mathrm{SL}_2(\mathbb{Z})$). This gives the LHS and the RHS. For k odd, consider $g(\tau) = f^2(\tau)$, which has weight $2k$. Then apply the result in the even case and divide LHS and RHS by 2. ☐

Corollary 6.20

$$M_k(\Gamma) = 0 \qquad \textit{for } k < 0\,, \qquad\qquad M_0(\Gamma) = \mathbb{C}\,.$$

Two modular forms for Γ whose Fourier coefficients agree up to a sufficiently high index coincide.

We extend the slash operator to $\mathrm{GL}_2^+(\mathbb{Q})$ (the $+$ indicating positive determinant) by

$$(f|_k\alpha)(\tau) := \det(\alpha)^{k/2} j(\alpha,\tau)^{-k} f(\alpha\tau).$$

In this way, elements of the center $\left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right)$ act trivially (unless k is odd and $a<0$).

Example 6.21

- If $\Gamma' \subset \Gamma$ then $M_k(\Gamma) \subset M_k(\Gamma')$ and $S_k(\Gamma) \subset S_k(\Gamma')$.
- Let $d, N > 0$ in $\mathbb{Z}$ and let $f \in M_k(\Gamma_0(N))$. Then

  $$f(d\tau) = d^{-k/2} f|_k \left(\begin{smallmatrix} d & \\ & 1 \end{smallmatrix}\right)(\tau) \in M_k(\Gamma_0(dN)).$$

  Same for cusp forms and for $\Gamma_1(N)$.

The proof of Hecke's bound for the Fourier coefficients of cusp forms still holds:

Proposition 6.22

Let $f = \sum_{n=1}^{\infty} a_n q_N^n \in S_k(\Gamma)$ *be a cusp form of weight* k *for a congruence subgroup* Γ *of level* N*. Then*

1. $Im(\tau)^{k/2}|f(\tau)|$ *is bounded on* $\mathbb{H}$*. (In fact, this is how one can* define *cusp forms without referring to Fourier expansions.)*
2. *Hecke's Bound:*

   $$|a_n| \leq C n^{k/2} \qquad \text{for some } C > 0.$$

   (The Ramanujan-Petersson conjecture (Deligne) improves this to $O(n^{(k-1)/2})$*.)*