Evaluations of the areal Mahler measure and polylogarithms

Matilde Lalín
Joint with Subham Roy

Université de Montréal

matilde.lalin@umontreal.ca
http://www.dms.umontreal.ca/~mlalin

Polylogarithms, Cluster Algebras, and Scattering Amplitudes
Brin Mathematics Research Center
University of Maryland
September 11, 2023

Université de Montréal

Extending Mersenne's construction

Pierce (1918) A construction for finding large prime numbers.

$P(x) \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_j (x - \alpha_j)$$

$$\Delta_n = \prod_j (\alpha_j^n - 1) \in \mathbb{Z}$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Université de Montréal

What are the best polynomials?

D. H. Lehmer (1933) : To improve the chances of finding a large prime among the factors of Δ_n, one needs Δ_n that grows slowly.

$$\lim_{n\to\infty} \frac{|\Delta_{n+1}|}{|\Delta_n|} > 1, \text{ but close to 1.}$$

$$\lim_{n\to\infty} \frac{|r^{n+1}-1|}{|r^n-1|} = \begin{cases} |r| & \text{if } |r|>1, \\ 1 & \text{if } |r|<1. \end{cases}$$

Université de Montréal

Mahler measure

For

$$P(x) = a \prod_j (x - \alpha_j)$$

$$M(P) = |a| \prod_{|\alpha_j|>1} |\alpha_j|, \qquad \mathrm{m}(P) = \log|a| + \sum_{|\alpha_j|>1} \log|\alpha_j|.$$

Thus, we want,

$$M(P) > 1 \text{ but close, or } \mathrm{m}(P) > 0 \text{ but close.}$$

Université de Montréal

Kronecker's Lemma

Kronecker (1857)

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$\mathrm{m}(P) = 0 \iff P(x) = x^k \prod \Phi_{n_i}(x),$$

where Φ_{n_i} are cyclotomic polynomials.

Université de Montréal

Lehmer's Question

Lehmer (1933)

Given $\varepsilon > 0$, can we find a polynomial $P(x) \in \mathbb{Z}[x]$ such that $0 < \mathrm{m}(P) < \varepsilon$?

Conjecture: No.

$$\mathrm{m}(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 0.162357612\ldots$$

Conjecture: This polynomial is the best possible.

Reid (1933) The above polynomial is the Alexander polynomial of the $(-2,3,7)$-pretzel knot.

Lehmer's Question - particular families

$P \in \mathbb{C}[x]$ reciprocal iff

$$P(x) = \pm x^{\deg P} P\left(x^{-1}\right).$$

Breusch (1951), Smyth (1971)

$P \in \mathbb{Z}[x]$ nonreciprocal,

$$\mathrm{m}(P) \geq \mathrm{m}(x^3 - x - 1) = 0.2811995743\ldots.$$

$$\Delta_{127} = 3,233,514,251,032,733$$

Université de Montréal

Lehmer's Question - degree dependent bounds

Dobrowolski (1979)

If $P \in \mathbb{Z}[x]$ is monic, irreducible and noncyclotomic of degree d, then

$$M(P) \geq 1 + c\left(\frac{\log\log d}{\log d}\right)^3,$$

where c is an absolute positive constant.

Université de Montréal

Mahler measure of multivariable rational functions

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the (logarithmic) Mahler measure is:

$$\begin{aligned} \mathrm{m}(P) &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} \\ &= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})| d\theta_1 \ldots d\theta_n, \end{aligned}$$

where $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : |x_i| = 1\}$.

Jensen's formula gives

$$\mathrm{m}(P) = \log |a| + \sum_{|\alpha_j|>1} \log |\alpha_j| \quad \text{if} \quad P(x) = a \prod_j (x - \alpha_j)$$

$$M(P) := \exp(\mathrm{m}(P)).$$

Université de Montréal

Boyd–Lawton Theorem

Boyd (1981), Lawton (1983)

For $P \in \mathbb{C}(x_1, \dots, x_n)^\times$,

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} \mathrm{m}(P(x, x^{k_2}, \dots, x^{k_n})) = \mathrm{m}(P(x_1, \dots, x_n))$$

The Mahler measure of several variable polynomials does not say much new about Lehmer's Question.

Special values of L-functions – Smyth's results

- Smyth (1981)

$$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

$$\chi_{-3}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ -1 & \text{if } n \equiv -1 \pmod 3 \\ 0 & \text{if } n \equiv 0 \pmod 2 \end{cases}$$

-

$$\mathrm{m}(1+x+y+z) = \frac{7}{2\pi^2} \zeta(3)$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1$$

Université de Montréal

Special values of L-functions - families

L. (2006)

$\pi^2 \mathrm{m}\left(1+\left(\frac{1-y_1}{1+y_1}\right)\left(\frac{1-y_2}{1+y_2}\right)x\right)$	$7\zeta(3)$
$\pi^4 \mathrm{m}\left(1+\left(\frac{1-y_1}{1+y_1}\right)\dots\left(\frac{1-y_4}{1+y_4}\right)x\right)$	$62\zeta(5)+\frac{14\pi^2}{3}\zeta(3)$
$\pi^6 \mathrm{m}\left(1+\left(\frac{1-y_1}{1+y_1}\right)\dots\left(\frac{1-y_6}{1+y_6}\right)x\right)$	$381\zeta(7)+62\pi^2\zeta(5)+\frac{56\pi^4}{15}\zeta(3)$
$\pi \mathrm{m}\left(1+\left(\frac{1-y_1}{1+y_1}\right)x\right)$	$2L(\chi_{-4},2)$
$\pi^3 \mathrm{m}\left(1+\left(\frac{1-y_1}{1+y_1}\right)\dots\left(\frac{1-y_3}{1+y_3}\right)x\right)$	$24L(\chi_{-4},4)+\pi^2L(\chi_{-4},2)$
$\pi^5 \mathrm{m}\left(1+\left(\frac{1-y_1}{1+y_1}\right)\dots\left(\frac{1-y_5}{1+y_5}\right)x\right)$	$160L(\chi_{-4},6)+20\pi^2L(\chi_{-4},4)+\frac{3\pi^4}{4}L(\chi_{-4},2)$

Université de Montréal

Special values of L-functions - families

Rogers & Zudilin (2014)

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)=\frac{15}{4\pi^2}L(E_{15},2)=L'(E_{15},0)$$

$$X=-\frac{1}{xy}, \qquad Y=\frac{(y-x)(1+xy)}{2x^2y^2}$$

$$E_{15a8}: Y^2=X^3+\left(\frac{1^2}{4}-2\right)X^2+X$$

Université de Montréal

Why do we get special values of L-functions?

- Deninger (1997) (aided by numerical computations of Boyd) : When P has coefficients in $\mathbb{Q}$ and $\{P = 0\} \cap \mathbb{T}^n = \varnothing$, the Mahler measure of P can be interpreted as a Deligne period in a mixed motive.
- In favorable cases, this motive is integral and the motivic version of Beĭlinson conjectures predicts that

$$L_X'(0) \sim_{\mathbb{Q}^\times} \operatorname{reg}(\xi)$$

- Even if $\{P = 0\} \cap \mathbb{T}^n \neq \varnothing$, the above can be adapted by using Jensen's formula.
  Formulas involving $\zeta(n)$, $L(\chi, n)$, etc, are expected to come from Borel's theorem.

Université de Montréal

An algebraic integration for Mahler measure

$$P(x,y) = y + x - 1 \qquad X = \{P(x,y) = 0\}$$

$$\mathrm{m}(P) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log|y + x - 1| \frac{dx}{x}\frac{dy}{y}$$

By Jensen's formula:

$$= \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log^+ |1 - x| \frac{dx}{x} = \frac{1}{2\pi i} \int_{\gamma} \log|y| \frac{dx}{x} = -\frac{1}{2\pi i} \int_{\gamma} \eta(x,y)$$

where

$$\gamma = X \cap \{|x| = 1, |y| \geq 1\}$$

$$\eta(x,y) = \log|x| d i \arg y - \log|y| d i \arg x \qquad d \arg x = \mathrm{Im}\left(\frac{dx}{x}\right)$$

- $\eta(x,y) = -\eta(y,x)$
- $\eta(x_1 x_2, y) = \eta(x_1, y) + \eta(x_2, y)$

Université de Montréal

$$\eta(x, 1-x) = di D(x).$$

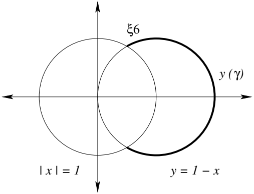
Bloch–Wigner dilogarithm

$$D(z) := \mathrm{Im}\left(\mathrm{Li}_2(z) + \log(1-z)\log|z|\right)$$

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-t)}{t} dt$$

$$\mathrm{m}(y+x-1) = -\frac{1}{2\pi i}\int_\gamma \eta(x,y) = -\frac{1}{2\pi} D(\partial\gamma)$$

$$= \frac{1}{2\pi}(D(e^{i\pi/3}) - D(e^{-i\pi/3})) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2).$$



Université de Montréal

General exact case

$P(x,y) \in \mathbb{Q}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi i} \int_\gamma \eta(x,y)$$

$$P(x,y) = P^*(x) y^{d_y} + \dots$$

We need

$$x \wedge y = \sum_j r_j \;\; z_j \wedge (1 - z_j) \quad \textit{in} \quad \bigwedge^2 (\mathbb{C}(X)^*) \otimes \mathbb{Q}$$

$(\{x,y\} = 0$ in $K_2(\mathbb{C}(X)) \otimes \mathbb{Q})$.

$$\int_\gamma \eta(x,y) = \sum r_j D(z_j)|_{\partial\gamma}.$$

Université de Montréal

When does it work? 2-variables

- $\eta(x, y)$ is exact, $\partial\gamma \neq \varnothing \rightsquigarrow$ Evaluation of Borel's regulator

$$\int_\gamma \eta(x, y) \rightsquigarrow D(z)$$

- γ cycle, other conditions $\rightsquigarrow$ Deligne period in a mixed motive, Beĭlinson's conjectures

$$\int_\gamma \eta(x, y) \rightsquigarrow L'(E, 0)$$

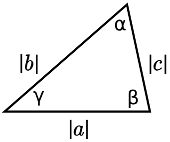
Université de Montréal

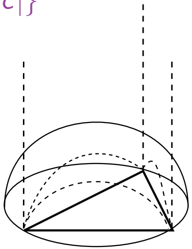
A connection with hyperbolic volumes

Cassaigne & Maillot (2000)

$a, b, c \in \mathbb{C}^*$, $a + bx + cy \in \mathbb{C}[x, y]$

$$\pi \mathrm{m}(a + bx + cy) = \begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle, \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle. \end{cases}$$





The areal Mahler measure

Pritsker (2008) $P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the (logarithmic) areal Mahler measure is:

$$\begin{aligned}\mathrm{m}_{\mathbb{D}}(P) =& \frac{1}{\pi^n} \int_{\mathbb{D}^n} \log |P(x_1, \ldots, x_n)| dA(x_1) \ldots dA(x_n) \\ =& \int_0^1 \cdots \int_0^1 \log |P(\rho_1 e^{2\pi i \theta_1}, \ldots, \rho_n e^{2\pi i \theta_n})| \rho_1 \cdots \rho_n \\ & d\rho_1 \ldots d\rho_n d\theta_1 \ldots d\theta_n,\end{aligned}$$

where $\mathbb{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : |x_1|, \ldots, |x_n| \leq 1\}$.
The natural measure in the A^0 Bergman space.
Pritsker (2008)

$$\mathrm{m}_{\mathbb{D}}(P) = \log |a| + \sum_{|\alpha_j|>1} \log |\alpha_j| + \frac{1}{2} \sum_{|\alpha_j|<1} (|\alpha_j|^2 - 1)$$

if $\quad P(x) = a \prod_{j=1}^d (x - \alpha_j)$.

Université de Montréal

Lehmer's Question

Pritsker (2008)

$$\mathrm{m}_{\mathbb{D}}(nx^n - 1) = \log n + \frac{n(n^{-2/n} - 1)}{2}$$

$$\mathrm{m}_{\mathbb{D}}(x^{2n} + nx^n + 1) = \log\left(\frac{n + \sqrt{n^2 - 4}}{2}\right) + \frac{n}{2}\left(\left(\frac{n - \sqrt{n^2 - 4}}{2}\right)^{2/n} - 1\right)$$

$\to 0$ as $n \to \infty$

Lehmer's Question has a negative answer!

Université de Montréal

The linear binomials

Pritsker (2008)

$$\mathrm{m}_{\mathbb{D}}(x+y) = -\frac{1}{4}.$$

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m + y) = \frac{1}{2^{m+1}} - \frac{1}{2}.$$

For $m, n \geq 2$,

$$\mathrm{m}_{\mathbb{D}}(x_1\cdots x_m + y_1\cdots y_n) = \frac{1}{4} + \binom{m+n-2}{m-1}\frac{1}{2^{m+n}}$$

$$-\frac{1}{2^{m+n}}\sum_{r=0}^{n-1}\binom{m+n-3-r}{m-2}2^r - \frac{1}{2^{m+n}}\sum_{r=0}^{m-1}\binom{m+n-3-r}{n-2}2^r$$

$$-\frac{m}{2^{m+n+1}}\sum_{r=0}^{n-1}\binom{m+n-1-r}{m}2^r - \frac{n}{2^{m+n+1}}\sum_{r=0}^{m-1}\binom{m+n-1-r}{n}2^r.$$

The linear trinomials

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}(1+x+y)=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)+\frac{1}{6}-\frac{11\sqrt{3}}{16\pi}.$$

Smith (1981)

$$\mathrm{m}(1+x+y)=\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2).$$

Université de Montréal

Ideas in the proof

$$\mathrm{m}_{\mathbb{D}}(1+x+y) = \frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}(|1+x|^2-1)dA(x) + \frac{1}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}}\log|1+x|dA(x).$$

If $x=\rho e^{i\theta}$

$$\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}(|1+x|^2-1)dA(x)$$

$$=\frac{1}{\pi}\int_{\frac{2\pi}{3}}^{\pi}\int_0^1(\rho^2+2\rho\cos\theta)\rho d\rho d\theta+\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}(\rho^2+2\rho\cos\theta)\rho d\rho d\theta$$

$$=-\frac{3\sqrt{3}}{16\pi}.$$

Université de Montréal

Ideas in the proof

$y = 1 + x$ and set $y = \rho e^{i\theta}$

$$\frac{1}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}} \log|1+x| dA(x)$$
$$= \frac{2}{\pi}\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta} (\log\rho)\rho d\rho d\theta$$
$$= \frac{1}{\pi}\int_0^{\frac{\pi}{3}}\left(4\cos^2\theta\log(2\cos\theta) - 2\cos^2\theta + \frac{1}{2}\right) d\theta.$$

$$\int_0^{\frac{\pi}{3}} \cos^2\theta\log(2\cos\theta)d\theta = \frac{1}{4}D(e^{i\pi/3}) + \frac{\pi}{12} - \frac{\sqrt{3}}{16}.$$

Université de Montréal

The linear trinomials

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}(\sqrt{2}+x+y) = \frac{L(\chi_{-4},2)}{\pi} + \mathcal{C}_{\sqrt{2}} + \frac{3}{8} - \frac{3}{2\pi},$$

where

$$\mathcal{C}_{\sqrt{2}} = \frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi^3}}\,{}_4F_3\left(\tfrac{1}{4},\tfrac{1}{4},\tfrac{3}{4},\tfrac{3}{4};\tfrac{1}{2},\tfrac{5}{4},\tfrac{5}{4};1\right) - \frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi^3}}\,{}_4F_3\left(\tfrac{3}{4},\tfrac{3}{4},\tfrac{5}{4},\tfrac{5}{4};\tfrac{3}{2},\tfrac{7}{4},\tfrac{7}{4};1\right),$$

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{z^n}{n!},$$

$$(a)_0 = 1, \qquad (a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

Cassaigne & Maillot (2000)

$$\mathrm{m}(\sqrt{2}+x+y) = \frac{L(\chi_{-4},2)}{\pi} + \frac{\log 2}{4}.$$

Université de Montréal

Another example

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}\left(y+\left(\frac{1-x}{1+x}\right)\right)=\frac{6}{\pi}L\left(\chi_{-4},2\right)-\log 2-\frac{1}{2}-\frac{1}{\pi}$$

Boyd (1992)

$$\mathrm{m}\left(y+\left(\frac{1-x}{1+x}\right)\right)=\frac{2}{\pi}L(\chi_{-4},2).$$

Université de Montréal

Higher Mahler measure

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the higher Mahler measure is

$$\mathrm{m}_h(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log^h |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

The higher areal Mahler measure is

$$\mathrm{m}_{\mathbb{D},h}(P) := \frac{1}{\pi^n} \int_{\mathbb{D}^n} \log^h |P(x_1, \ldots, x_n)| dA(x_1) \ldots dA(x_n).$$

Université de Montréal

Higher Mahler measure

Sasaki (2015)

$$\mathrm{m}_h\left(\frac{1-x}{1+x}\right) = \frac{|E_h|\pi^h}{2^h} \quad h \text{ even}, \qquad \text{and} \qquad = 0 \quad h \text{ odd}.$$

where E_n denotes the nth Euler number,

$$\frac{s}{e^s-1} = \sum_{n=0}^{\infty} B_n \frac{s^n}{n!} \qquad \frac{2e^s}{e^{2s}+1} = \sum_{n=0}^{\infty} E_n \frac{s^n}{n!}$$

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!} \quad \text{and} \quad L(\chi_{-4}, 2n+1) = \frac{(-1)^n E_{2n}\pi^{2n+1}}{2^{2n+2}(2n)!}.$$

Université de Montréal

Higher Mahler measure

L. & Roy (2024) For $h \in \mathbb{Z}_{>0}$ even

$$\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = -2(h-1)B_h(\pi i)^{h-1} + \frac{E_h(\pi i)^h}{2^h} - \frac{E_{h-2}(\pi i)^{h-2}h(h-1)}{2^{h-2}}\log 2$$
$$-\frac{4h!}{2^h}\sum_{m=2}^{h-1}(1-2^{1-m})\zeta(m)\frac{E_{h-m-1}(\pi i)^{h-m-1}}{(h-m-1)!}$$
$$-2(\pi i)^{h-1}\sum_{m=0}^{h}\binom{h}{m}(1-2^{1-m})(1-2^{1-h+m})B_m B_{h-m}.$$

For h odd, $\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = 0.$

Université de Montréal

The transformation $x \mapsto x^r$

$$P(\mathbf{x}) = \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{C}[x_1, \dots, x_n] \qquad \mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n},$$

Let $A \in M(n \times n, \mathbb{Z})$, $\det(A) \neq 0$.

$$P^{(A)}(\mathbf{x}) := \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{A\mathbf{m}}.$$

Then

$$\mathrm{m}(P) = \mathrm{m}\left(P^{(A)}\right).$$

Particular case: $x \mapsto x^r$.

Université de Montréal

The transformation $x \mapsto x^r$ and a limiting result

L. & Roy (2023++) $r, s \in \mathbb{Z}_{>0}$,

$$\mathrm{m}_{\mathbb{D}}(x^r - a) = \begin{cases} \log^+ |a| & |a| \geq 1, \\ \frac{r}{2}\left(|a|^{\frac{2}{r}} - 1\right) & |a| \leq 1. \end{cases} \qquad \mathrm{m}_{\mathbb{D}}(x^r + y^s) = -\frac{rs}{2(r+s)}.$$

L. & Roy (2023++) Let $P(x_1, \dots, x_n) \in \mathbb{C}(x_1, \dots, x_n)^\times$ and let $P(0, x_2, \dots, x_n) \in \mathbb{C}(x_2, \dots, x_n)^\times$ be the rational function resulting from P by setting $x_1 = 0$. Let $r \in \mathbb{Z}_{>0}$. Then

$$\lim_{r \to \infty} \mathrm{m}_{\mathbb{D}}(P(x_1^r, x_2, \dots, x_n)) = \mathrm{m}_{\mathbb{D}}(P(0, x_2, \dots, x_n)).$$

Université de Montréal

Looking ahead

- $\mathrm{m}_{\mathbb{D}}(1+x^r+y^s)$ and $\mathrm{m}_{\mathbb{D}}((1+x)^r+y^s)$ (L. & Roy, in progress)
- A regulator explanation for nice areal Mahler measure formulas.
- Areal Mahler measure for elliptic curve cases.

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)=\frac{15}{4\pi^2}L(E_{15},2)$$

- Non-trivial examples of generalized areal Mahler measure, higher areal Mahler measure, zeta Mahler measure.
- Study more general transformations.
- Function field analogue (Roy, in progress).

Université de Montréal

Thanks for your attention!

Université de Montréal

Generalized Mahler measure

$P_1, \dots, P_r \in \mathbb{C}(x_1, \dots, x_n)^\times$, the generalized (logarithmic) Maher measure is

$$\mathrm{m}_{\max}(P_1, \dots, P_r) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \max\{\log|P_1|, \dots, \log|P_r|\} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

The generalized (logarithmic) areal Mahler measure is

$$\mathrm{m}_{\mathbb{D},\max}(P_1, \dots, P_r) = \frac{1}{\pi^n} \int_{\mathbb{D}^n} \max\{\log|P_1|, \dots, \log|P_r|\} dA(x_1) \dots dA(x_n).$$

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D},\max}(x_1, \dots, x_n) = -\frac{1}{2n}.$$

Université de Montréal

Zeta Mahler measure

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the zeta Mahler measure is

$$Z(s, P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} |P(x_1, \ldots, x_n)|^s \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

$$Z(s, P) = \sum_{k=0}^{\infty} \frac{\mathrm{m}_k(P) s^k}{k!}.$$

The areal zeta Mahler measure is

$$Z_{\mathbb{D}}(s, P) := \frac{1}{\pi^n} \int_{\mathbb{D}^n} |P(x_1, \ldots, x_n)|^s \, dA(x_1) \ldots dA(x_n).$$

Université de Montréal

An example of Zeta Mahler measure

L. & Roy (2024)

$$Z_{\mathbb{D}}(s, x+1) := \frac{1}{\pi}\int_{\mathbb{D}} |x+1|^s dA(x) = \exp\left(\sum_{j=2}^{\infty} \frac{(-1)^j}{j}(1-2^{1-j})(\zeta(j)-1)s^j\right)$$

Akatsuka (2009)

$$Z(s, x+1) := \frac{1}{2\pi}\int_{\mathbb{T}} |x+1|^s \frac{dx}{x} = \exp\left(\sum_{j=2}^{\infty} \frac{(-1)^j}{j}(1-2^{1-j})\zeta(j)s^j\right).$$

$$Z_{\mathbb{D}}(s, x+1) = \frac{s+1}{(s/2+1)^2} Z(s, x+1).$$

Université de Montréal