Dilogarithm identities, cluster algebras, and cluster scattering diagrams

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Plan of Talk

In this talk, we review the relation between dilogarithm identities (DI) and cluster algebras (CA), which is recently updated in view of cluster scattering diagrams (CSD).

Caution: Cluster scattering diagrams are nothing to do with scattering amplitudes which is one of the theme of this workshop.

Dilogarithms

Euler dilogarithm: $(x \leq 1)$ (1768)

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2} = -\int_0^x \frac{\log(1-y)}{y}\,dy.$$

Rogers dilogarithm: $(0 \leq x \leq 1)$ (1907)

$$L(x) = -\frac{1}{2}\int_0^x \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} dy. = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x).$$

modified Rogers dilogarithm (no official name): $(0 \leq x)$ (1990's–)

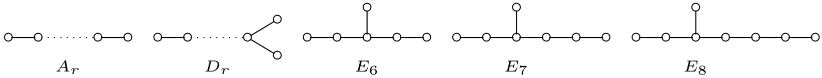
$$\tilde{L}(x) = \frac{1}{2}\int_0^x \left\{ \frac{\log(1+y)}{y} - \frac{\log y}{1+y} \right\} dy. = -\mathrm{Li}_2(-x) - \frac{1}{2}\log x \log(1+x) = L\left(\frac{x}{1+x}\right).$$

In this talk we maily use $\tilde{L}(x)$. (Its importance is a key point of this talk.)

Dilogarithm conjecture from Bethe ansatz method

- In 1980's Faddeev and others in Leningrad (St. Petersburg) started to study integrable systems by the Bethe ansatz method.
- The Rogers dilogartihm $L(x)$ mysteriously appeared through the calculation of the specific heats of various integrable lattice models.

X_r: simply laced Dynkin diagram:



For nodes a and b in X_r, we write $a \sim b$ if it is adjacent in X_r.
Fix an integer $\ell \geq 2$, called the level.
For a pair (X_r, ℓ), we define a system of algebraic equations for $Q_m^{(a)}$ $(a = 1, \dots, r;\ m = 1. \dots, \ell - 1)$:

$$(Q\text{-system}) \quad {Q_m^{(a)}}^2 = Q_{m+1}^{(a)} Q_{m-1}^{(a)} + \prod_{b : b \sim a} Q_m^{(b)}, \quad Q_0^{(a)} = Q_\ell^{(a)} = 1.$$

Conjecture [Kirillov89, Bazhanov-Reshetikhin 90]

For the unique positive real solution of the Q-system, the following equality holds:

$$\sum_{a=1}^{r} \sum_{m=1}^{\ell-1} L\left(\frac{\prod_{b:b\sim a} Q_m^{(b)}}{{Q_m^{(a)}}^2} \right) = \frac{rh(\ell-1)}{h+\ell} \frac{\pi^2}{6} \quad (h\text{: Coxeter number of } X_r).$$

Remarkably, Kirillov proved it for type A_r by the explicit solution [Kirillov89].

Functional generalization of dilogarithm conjecture

- The Y-system (a system of functional equations) was introduced by Al. Zamolodchikov in 1991 to study some integrable field theories.
- Gliozzi and Tateo conjectured the functional generalization of the dilogarithm conjecture based on the Y-system for certain integrable field theories.

For the same pair (X_r, ℓ) of the Q-system, we define a system of functional equations for $Y_m^{(a)}(u)$ ($a = 1, \dots, r$; $m = 1. \dots, \ell - 1$; $u \in \mathbb{Z}$):

$$(Y\text{-system}) \quad Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{\prod_{b:b\sim a}(1 + Y_m^{(b)}(u)),}{(1 + Y_{m+1}^{(a)}(u)^{-1})(1 + Y_{m-1}^{(a)}(u)^{-1})},$$

$$Y_0^{(a)}(u)^{-1} = Y_\ell^{(a)}(u)^{-1} = 0.$$

One can regard it as a system of recursion relations along discrete parameter u (discrete dynamical system) with the initial variables $Y_m^{(a)}(0)$ and $Y_m^{(a)}(1)$.

Conjecture [Gliozzi-Tateo95]

(1) (Periodicity) $Y_m^{(a)}(u + 2(h + \ell)) = Y_m^{(a)}(u)$. (for $\ell = 2$, [Zamlodchikov91])
(2) (functional dilogarithm identity)

$$\sum_{u=0}^{2(h+\ell)-1} \sum_{a=1}^{r} \sum_{m=1}^{\ell-1} \tilde{L}(Y_m^{(a)}(u)) = 2rh(\ell - 1)\frac{\pi^2}{6}.$$

For the positive constant solution, the DI reduces to the DI conjectured by [BR90].

Examples of Y-system DI

Example 1. $(X_r, \ell) = (A_1, 2)$, where $h = 2$, period $2(h+\ell) = 8$.

We have only variables $Y(u) = Y_1^{(1)}(u)$, and the Y-system is given by

$$Y(u+1)Y(u-1) = 1.$$

It has a reduced period of 4: $Y(u+4) = Y(u+2)^{-1} = Y(u)$. The corresponding DI is

$$\tilde{L}(y) + \tilde{L}(y^{-1}) = \frac{\pi^2}{6}.$$

This is Euler's identity.

Example 2. $(X_r, \ell) = (A_2, 2)$, where $h = 3$, period $2(h+\ell) = 10$.

We have variables $Y_1(u) := Y_1^{(1)}(u)$, $Y_2(u) := Y_1^{(2)}(u)$, and the Y-system is given by

$$Y_1(u+1)Y_1(u-1) = 1 + Y_2(u), \quad Y_2(u+1)Y_2(u-1) = 1 + Y_1(u).$$

It has a half periodicity $Y_1(u+5) = Y_2(u)$, $Y_2(u+5) = Y_1(u)$. The corresponding DI is

$$\tilde{L}(y_1) + \tilde{L}(y_2(1+y_1)) + \tilde{L}(y_1^{-1}(1+y_2+y_1y_2)) + \tilde{L}(y_1^{-1}y_2^{-1}(1+y_2)) + \tilde{L}({y_2}^{-1}) = \frac{\pi^2}{2}.$$

This is Abel's identity (the pentagon identity).

So, Y-system DIs are vast generalization of these classic identities by root systems. They were very mysterious and only proved partially before cluster algebras (= B.C.).

1. History in B.C. (1980s–2000)
2. DI and CA (2000–2015)
3. DI and CSD (2015–)

Development after cluster algebra

Solutions of Y-system conjectures for (X_r, ℓ):

Who and When	periodicity	DI	idea/method/result
Gliozzi-Tateo 95	$(A_r, 2)$	$(A_r, 2)$	explicit solution
Frenkel-Szenes 95	$(A_r, 2)$	$(A_r, 2)$	explicit solution constancy condition (1)
Fomin-Zelevinsky 00∼	–	–	cluster algebra
Fomin-Zelevinsky 03	(any, 2)		cluster structure (2) Coxeter transformation (3)
Chapoton 05		(any, 2)	(1) + (2) evaluation at $0/\infty$ limit (4)
Szenes 06 Volkov 06	$(A_r,$ any$)$		flat connection on graph explicit solution
Fomin-Zelevinsky 07	–	–	coefficients/ F-polynomials (5)
Keller 08	(any, any)		(5) cluster category Auslander-Reiten theory
N 09		(any, any)	(1)+(2)+(3)+(4)+(5)

There are other types of Y-systems, and the corresponding problems were also solved by the cluster algebraic methods.
nonsimply-laced Y-system: [Inoue-Iyama-Kuniba-N-Suzuki13]
sine-Gordon Y-system: [N-Tateo10], [N-Stella14]

Cluster algebra basics (1)

We say that an $r \times r$ integer matrix $B = (b_{ij})$ is skew-symmetrizable if it has a decomposition (skew-symmetric decomposition)

$$B = \Delta\Omega,$$

where Δ is a diagonal matrix whose diagonals are positive integers and Ω is a skew-symmetric matrix.
For an integer a, let

$$[a]_+ = \max(a, 0).$$

For an $n \times n$ skew-symmetrizable matrix B and $k = 1, \dots, r$, a new $r \times r$ integer matrix $B' = (b'_{ij}) = \mu_k(B)$ is defined by

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} + b_{ik}[b_{kj}]_+ [-b_{ik}]_+ b_{kj} & i, j \neq k. \end{cases}$$

It is called the mutation of B in direction k.

Fact

(1) B' is also skew-symmetrizable with common skew-symmetrizer Δ.
(2) μ_k is involutive, i.e., $\mu_k(B') = B$.

Cluster algebra basics (2)

A pair $\Upsilon = (\mathbf{y}, B)$ is called a Y-seed, where $\mathbf{y} = (y_1, \dots, y_r)$ is an r-tuple of formal variables, and B is an $r \times r$ skew-symmetrizable matrix.
For a Y-seed $\Upsilon = (\mathbf{y}, B)$ and $k = 1, \dots, r$, a new Y-seed $\mu_k(\Upsilon) = \Upsilon' = (\mathbf{y}', B')$ is defined by $B' = \mu_k(B)$ and

$$y'_i = \begin{cases} y_k^{-1} & i = k, \\ y_i y_k^{[b_{ki}]_+}(1 + y_k)^{-b_{ki}} & i \neq k. \end{cases}$$

It is called the mutation of Υ in direction k.

Fact

μ_k is involutive, i.e., $\mu_k(\mathbf{y}', B') = (\mathbf{y}, B)$.

We define a left action of permutation σ of $\{1, \dots, r\}$ on a seed $\Upsilon = (\mathbf{y}, B)$ by $\sigma(\Upsilon) = \Upsilon' = (\mathbf{y}', B')$, where

$$y'_i = y_{\sigma^{-1}(i)}, \quad b'_{ij} = b_{\sigma^{-1}(i)\sigma^{-1}(j)}.$$

DI associated with a period in CA

Consider a sequence of mutations

$$\Upsilon(0) = (\mathbf{y}(0), B(0)) \xrightarrow{k_0} \Upsilon(1) = (\mathbf{y}(1), B(1)) \xrightarrow{k_1} \cdots \xrightarrow{k_{P-1}} \Upsilon(P) = (\mathbf{y}(P), B(P)).$$

We say that it is σ-period if $\Upsilon(P) = \sigma(\Upsilon(0))$ for a permutation σ.

- After proving several Y-system DIs, I recognized that the periodicity is essential.

Theorem [N12]. (DI associated with a period in CA)

For any σ-period as above, the following DI holds:

$$\sum_{s=0}^{P-1} \delta_{k_s} \tilde{L}(y_{k_s}(s)) = N\frac{\pi^2}{6},$$

where $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_r)$ is a common skew symmetrizer of $B(s) = \Delta\Omega(s)$ and N is some positive integer. It is also rewritten in the form (zero constant form)

$$\sum_{s=0}^{P-1} \varepsilon_s \delta_{k_s} \tilde{L}(y_{k_s}(s)^{\varepsilon_{k_s}}) = 0,$$

where $\varepsilon_s \in \{\pm 1\}$ is the tropical sign of $y_{k_s}(s)$.

Y-systems are embedded in some sequences of mutations. Their periodicities and DIs are special instances of the above.

Examples of DIs (1)

type A_1 (involution). $r = 1$, $B = (0)$.

By the involution of the mutation, we have a periodicity

$$\Upsilon(0) = (\mathbf{y}, B) \xrightarrow{1} \Upsilon(1) \xrightarrow{1} \Upsilon(0).$$

The associated DI is

$$\tilde{L}(y_1) + \tilde{L}(y_1^{-1}) = \frac{\pi^2}{6}.$$

This is Euler's identity. The zero constant form is trivial:

$$\tilde{L}(y_1) - \tilde{L}(y_1) = 0.$$

type $A_1 \times A_1$ (commutativity/square periodicity). $r = 2$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Since two mutations μ_1 and μ_2 are commutative, we have a periodicity

$$\Upsilon(0) = (\mathbf{y}, B) \xrightarrow{1} \Upsilon(1) \xrightarrow{2} \Upsilon(2) \xrightarrow{1} \Upsilon(3) \xrightarrow{2} \Upsilon(0).$$

The associated DI is

$$\tilde{L}(y_1) + \tilde{L}(y_2) + \tilde{L}(y_1^{-1}) + \tilde{L}(y_2^{-1}) = \frac{\pi^2}{3}.$$

Again, this is Euler's identity. The zero constant form is trivial:

$$\tilde{L}(y_1) + \tilde{L}(y_2) - \tilde{L}(y_1) - \tilde{L}(y_2) = 0.$$

Examples of DIs (2)

type A_2 (pentagon periodicity). $r = 2$, $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We have a nontrivial periodicity

$$\Upsilon(0) = (\mathbf{y}, B) \xrightarrow{1} \Upsilon(1) \xrightarrow{2} \Upsilon(2) \xrightarrow{1} \Upsilon(3) \xrightarrow{2} \Upsilon(4) \xrightarrow{1} \tau_{12}\Upsilon(0).$$

The associated DI is

$$\tilde{L}(y_1)+\tilde{L}(y_2(1+y_1))+\tilde{L}(y_1^{-1}(1+y_2+y_1y_2))+\tilde{L}(y_1^{-1}y_2^{-1}(1+y_2))+\tilde{L}(y_2{}^{-1}) = \frac{\pi^2}{2}.$$

This is Abel's identity (the pentagon identity). The zero constant form is

$$\tilde{L}(y_1) + \tilde{L}(y_2(1 + y_1)) - \tilde{L}(y_1(1 + y_2 + y_1y_2)^{-1}) - \tilde{L}(y_1y_2(1 + y_2)^{-1}) - \tilde{L}(y_2) = 0.$$

type B_2 (hexagon periodicity). $r = 2$, $B = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \Delta\Omega$.

We have a nontrivial periodicity

$$\Upsilon(0) = (\mathbf{y}, B) \xrightarrow{1} \Upsilon(1) \xrightarrow{2} \Upsilon(2) \xrightarrow{1} \Upsilon(3) \xrightarrow{2} \Upsilon(4) \xrightarrow{1} \Upsilon(5) \xrightarrow{2} \Upsilon(0).$$

The associated DI in the zero constant form is

$$\begin{aligned}&\tilde{L}(y_1) + 2\tilde{L}(y_2(1 + y_1)) - \tilde{L}(y_1(1 + y_2 + y_1y_2)^{-2})\\ &- 2\tilde{L}(y_1y_2(1 + 2y_2 + y_2^2 + y_1y_2^2)^{-1}) - \tilde{L}(y_1y_2^2(1 + y_2)^{-2}) - 2\tilde{L}(y_2) = 0.\end{aligned}$$

Methods/Ideas of Proof of DIs in CA

Method 1: Algebraic method [N12].

Constancy condition [Frenkel-Szenes95] (based on the idea of [Bloch78]:

$$\sum_{t=1}^{P} f_t(u) \wedge (1 + f_t(u)) = 0 \quad \Longrightarrow \quad \sum_{t=1}^{P} \tilde{L}(f_t(u)) = \text{const.}$$

To show the constancy condition, we use the idea of [Fock-Goncharov09]):
For each Y-seed $\Upsilon(s)$, we attach certain quantity $V(s)$ such that $V(s+1) - V(s) = \delta_{k_s} y_{k_s}(s) \wedge (1 + y_{k_s}(s))$. Then, the periodicity implies the constancy condition. (The proof does not explain why such $V(s)$ exists.)

Method 2: via Quantization [Kashaev-N11].

Fo each σ-period one obtains the quantum dilogarithm identities (QDI) for Faddeev's quantum dilogarithm $\Phi_q(x)$ [Fock-Goncharov09]. Taking the limit $q \to 1$ and apply the saddle point method, we recover the classical DI. (The saddle point method (in multivariables) is standard in physics, but difficult to be validated rigorously.)

Method 3: Classical mechanical method [Gekhtman-N-Rupel17].

One can bypass quantization by directly formulating mutations as classical mechanical system, where the Hamiltonian is given by the Euler dilogarithm [Fock-Goncharov09]. Then, the modified Rogers dilogarithm appears as the Lagrangian, and the DI is obtained as the invariance of the action integral due to the discrete-time analogue of Noether's theorem. (This explains the intrinsic meaning of DIs.)

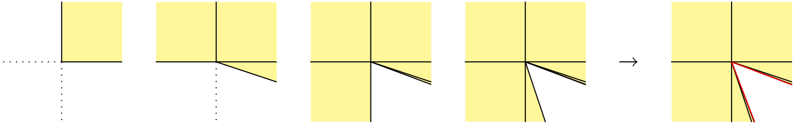
Cluster Scattering Diagram (CSD)

- Around 2015, Gross-Hacking-Keel-Kontsevich [GHKK18] proved some important conjectures on cluster algebras by using cluster scattering diagrams (CSDs).
- The notion of scattering diagram (a.k.a. wall-crossing structure) was originally introduced by [Gross-Siebert11] and [Kontsevich-Soibelman06] to study the homological mirror symmetry.
- Roughly speaking, any cluster pattern is embedded in the corresponding CSD.

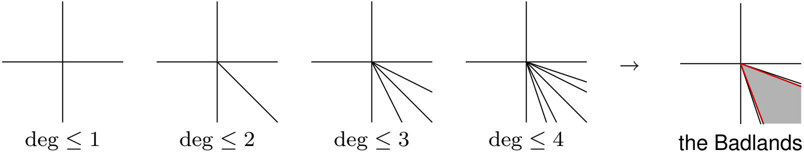
Example:

$$B = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} \quad \text{infinite type, nonaffine}$$

G-fan (representing a cluster pattern). principle: mutation



CSD (only the support is presented). principle: consistency



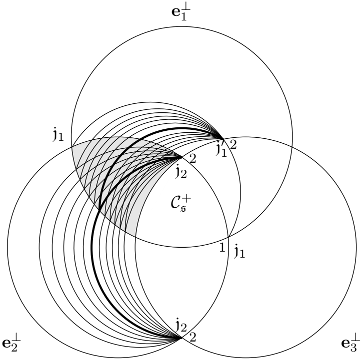
Badlands (the dark side)

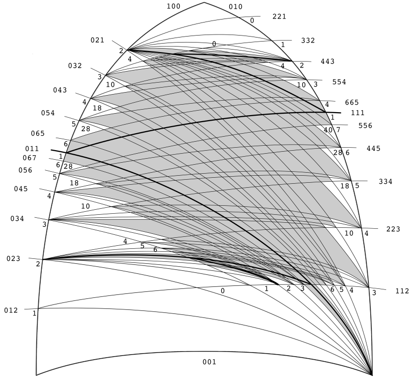
Badlands National Park, South Dakota, USA

Example: the Badlands in a rank 3 CSD

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \quad \text{(infinite, nonaffine).}$$

the stereo graphic projection of the support: (The right figure is the magnified one of the shaded region in the left figure.)





[N23] T. Nakanishi, *Cluster algebras and scattering diagrams*, MSJ Mem. 41 (2023), 279 pp.

CSD Basics

- $B = \Delta\Omega$: skew-symmetric decomposition of the initial exchange matrix
- the structure group $G = G_\Omega$
  a lattice $N = \mathbb{Z}^r$, $N^+ =: \{n \in N \mid n \neq 0, n \in (\mathbb{Z}_{\geq 0})^r\}$.
  Lie algebra $\mathfrak{g}$: generators X_n $(n \in N^+)$ with $[X_n, X_{n'}] = \{n, n'\}_\Omega X_{n+n'}$.
  $\overline{\mathfrak{g}}$: completion of $\mathfrak{g}$ with respect to $\deg$
  exponential group $G = \exp(\overline{\mathfrak{g}})$: the product is defined by the Baker-Campbell-Hausdorff formula
- dilogarithm elements: $\Psi[n] = \exp(\sum_{j=1}^\infty \frac{(-1)^{j+1}}{j^2} X_{jn}) \in G$ $(n \in N^+)$.
- action of G on $\mathbb{Q}[[y]]$: $X_n(y^{n'}) = \{n, n'\}_\Omega y^{n+n'}$. $\Psi[n]y^{n'} = y^{n'}(1+y^n)^{\{n,n'\}_\Omega}$.
- pentagon relation: if $\{n, n'\} = c > 0$,
  $$\Psi[n]^{1/c}\Psi[n']^{1/c} = \Psi[n']^{1/c}\Psi[n+n']^{1/c}\Psi[n]^{1/c}.$$
- wall $\mathbf{w} = (\mathfrak{d}, \Psi[n]^c)_n$: $n \in N^+$: normal vector,
  codimension 1 cone $\mathfrak{d} \subset n^\perp \subset \mathbb{R}^r$: support, $\Psi[n]^c$ $(c \in \mathbb{Q})$: wall element
- scattering diagram $\mathfrak{D}$: a collection of walls (satisfying the finiteness condition)
- scattering diagram $\mathfrak{D}$ is consistent if for any loop γ in $\mathbb{R}^r$, the product of wall elements (with intersection sign) along γ is the identity in G.

Theorem/Definition [GHKK18] Cluster scattering diagram (CSD)

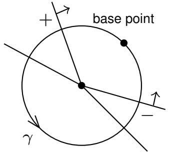
There is a unique consistent scattering diagram $\mathfrak{D}$ (up to equivalence) such that
- $(e_i^\perp, \Psi[e_i]^{\delta_i})_{e_i}$ $(i = 1, \dots, r)$ are walls of $\mathfrak{D}$ (incoming walls)
- any other wall $\mathbf{w} = (\mathfrak{d}, \Psi[n]^c)_n$ in $\mathfrak{D}$ satisfies $Bn \notin \mathfrak{d}$ (outgoing walls)

DI in CSD

$\mathfrak{D} = \mathfrak{D}(B)$: CSD for the initial exchange matrix B
γ: any loop in $\mathfrak{D}$
• consistency relation along a loop γ:

$$\overleftarrow{\prod_s} \Psi[n_s]^{\epsilon_s c_s} = \mathrm{id}$$

ϵ_s: the intersection sign, $c_s \in \mathbb{Q}$.



Theorem [N21]

The following DI holds:

$$\sum_s \epsilon_s c_s \tilde{L}(y_s) = 0,$$

$$y_s = \Big(\overrightarrow{\prod_{t:t<s}} \Psi[n_t]^{-\epsilon_t c_t}\Big) y^{n_s} \quad \text{(generalization of mutation).}$$

- The sum is an infinite one in general.
- When the loop γ is completely inside the G-fan, the DI coincides with the one associated with a period of CA.
- The proof is given by the extension of Method 3 (classical mechanical method).

Infinite reduciblity

The following structure theorem for CSDs is known.

Fact [GHKK18,N23]

Any consistency relation in a CSD is reduced to a trivial one by applying the pentagon and commutative relation in G possibly infinitely many times.

Shortly speaking, the dilogarithm elements and the pentagon relation are everything for a CSD.
As a result, we have the following infinite reducibility of DI for a CSD.

Theorem [N21] (inifnite reducibility of Di)

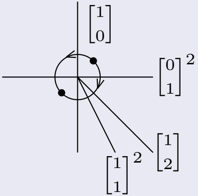
The DI associated with any loop in a CSD is reduced to a trivial one by applying pentagon identity possibly infinitely many times.

This is also applicable to the DI associated with any period in a CA, which is a finite sum.
On the other hand, according to the recent result of [de Jeu20], any finite DI whose arguments are rational functions is finitely reducible.
Thus, the DI associated with any period in a CA is actually finitely reducible. (This is a little disappointing to me at this moment because the structure group G fails to catch this finite reducibility.)

Examples (1)

type B_2 (hexagon periodicity). $r = 2$, $B = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \Delta\Omega$.

We write $[n] := \Psi[n]$. The consistency relation along γ is generated by the pentagon relation as follows:



$$\begin{aligned}
\begin{bmatrix}0\\1\end{bmatrix} \underbrace{\begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}} &= \underbrace{\begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix}} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \\
&= \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} \underbrace{\begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix}} \begin{bmatrix}0\\1\end{bmatrix} \\
&= \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}1\\2\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}.
\end{aligned}$$

Accordingly, one can generate the corresponding DI by the pentagon identity.

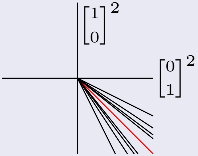
$$\begin{aligned}
&\underbrace{\begin{bmatrix}1\\0\end{bmatrix}^{-1} \begin{bmatrix}0\\1\end{bmatrix}^{-1}} \tilde{L}(y^{e_2}) + \underline{\begin{bmatrix}1\\0\end{bmatrix}^{-1} \tilde{L}(y^{e_2})} + \underline{\tilde{L}(y^{e_1})} \\
&= \begin{bmatrix}0\\1\end{bmatrix}^{-1} \begin{bmatrix}1\\1\end{bmatrix}^{-1} \underline{\begin{bmatrix}1\\0\end{bmatrix}^{-1} \tilde{L}(y^{e_2})} + \begin{bmatrix}0\\1\end{bmatrix}^{-1} \begin{bmatrix}1\\1\end{bmatrix}^{-1} \underline{\tilde{L}(y^{e_1})} + \begin{bmatrix}0\\1\end{bmatrix}^{-1} \tilde{L}(y^{(1,1)}) + \tilde{L}(y^{e_2}) \\
&= \cdots
\end{aligned}$$

So, this is finitely reducible.

Examples (2)

type $A_1^{(1)}$ (infinite periodicity). $r=2$, $B=\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\Delta\Omega$.

The consistency relation along γ is as follows:

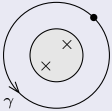


$$\begin{bmatrix}0\\1\end{bmatrix}^2\begin{bmatrix}1\\0\end{bmatrix}^2=\begin{bmatrix}1\\0\end{bmatrix}^2\begin{bmatrix}2\\1\end{bmatrix}^2\begin{bmatrix}3\\2\end{bmatrix}^2\cdots\prod_{j=0}^{\infty}\begin{bmatrix}2^j\\2^j\end{bmatrix}^{2^{2-j}}\cdots\begin{bmatrix}2\\3\end{bmatrix}^2\begin{bmatrix}1\\2\end{bmatrix}^2\begin{bmatrix}0\\1\end{bmatrix}^2$$

The associated DI is an infinite sum and infinitely reducible.

CA associated with torus with two punctures

There is period of length 32 that is not a product of the pentagon and square periodicity [Fomin-Shapiro-Thurston07]. Similar examples are known in [Kim-Yamazaki18]. A schematic picture in CSD is as follows:



The loop γ is not shrinkable inside the G-fan due to an obstacle (joint of type $A_1^{(1)}$). The associated DI is infinitely reducible. (However, according to the result of [de Jeu20], this is actually finitely reducible by some other means.)