Daniil Rudenko

Goncharov's Programme

"Old" story connecting polylogs & alg K-thy

Newer: Cluster alg's

well understood

Mixed motives

Polylogs

Cluster algebras

(e.g. in MM sometimes can be proved w. help of cl. alg.)

---

multiple polylogs, fct of many var's ($\mathbb{C}x$)

$$\mathrm{Li}_{n_1,\dots,n_k}(a_1,\dots,a_k) = \sum_{0<m_1<\dots<m_k} \frac{a_1^{m_1}\cdots a_k^{m_k}}{m_1^{n_1}\cdots m_k^{n_k}} \qquad |a_i|<1$$

$n_1,\dots,n_k \in \mathbb{N}$

· $\mathrm{Li}_2(a)$ (Euler) dilog multivaluedness

"ignore everything of the form $\pi \cdot *$"

Depth reduction ↔ kind of Hilbert 13th prob

Shuffle el's: combinatorics of power series

Li_{11} .. also $\mathrm{Li}_{11}(a_1,a_2) = \mathrm{Li}_2\left(\left[\frac{1-a_1}{1-a_2^{-1}}\right] - \left[\frac{1}{1-a_2}\right] - (a_1 a_2)\right)$

together → 5-term (Don's version) 5-fold sym

→ Relation to cluster algebras $\equiv \frac{\pi^2}{6} - \log(a)\log(1-a) + \log\left(\frac{1-a}{1-ab}\right)\log\left(\frac{1-b}{1-ab}\right) - \mathrm{Li}_j(b)\log(a b)$

Jump: assume we know all the relations...

Then build a Hopf alg.

Let F be an arbitrary field.

Then for every $n \geq 0$ define

$$\mathcal{H}_n(F) \underset{\mathbb{Q}\text{-vs}}{=} \left\langle \begin{matrix} \mathbb{Q}\text{-span of symbols} \\ \mathrm{Li}^{\mathcal{M}}_{n_1,\dots,n_k}(a_1,\dots,a_k), \; a_1,\dots,a_k \in F \end{matrix} \right\rangle \Big/ \begin{matrix}\text{equations} \\ \text{for polylogs} \\ \text{evaluated at } a_i \in F\end{matrix}$$

$$\mathcal{H}_0(F) = \mathbb{Q}$$

$$\mathcal{H}(F) = \bigoplus_{n \geq 0} \mathcal{H}_n \quad \Bigg| \quad \mathcal{L} = \mathcal{H}_{>0} \Big/ \mathcal{H}_{>0} \cdot \mathcal{H}_{>0}$$

What does this constr. give for $n=1$?

$$\mathcal{H}_1(F) = \langle \mathbb{Q}\text{-span of } \log^{\mathcal{M}}(a) := -\mathrm{Li}_1^{\mathcal{M}}(1-a) \rangle \Big/ \langle \log^{\mathcal{M}}(a) + \log^{\mathcal{M}}(b) = \log^{\mathcal{M}}(ab) \rangle$$

$$\mathcal{L}_1 = \mathcal{H}_1(F)$$

$$= F^\times \otimes \mathbb{Q} = K_1(F) \otimes \mathbb{Q}$$

for $\mathcal{H}_2(F)$ or rather its $\mathcal{L}$

$$\left(B_2(F) = \langle \mathbb{Q}[F^\times]\text{-span of } \mathrm{Li}_2^{\mathcal{M}}(a) \rangle \right) \Big/ \quad \rightsquigarrow \text{kill lower order polylogs}$$

$$0 \to \mathcal{H}_1 \cdot \mathcal{H}_1 \to \mathcal{H}_2 \to \mathcal{L}_2 \to 0$$

Can prove $\mathcal{L}_2(F) = B_2(F) = \dfrac{\mathbb{Q}[\mathrm{Li}_2^{\mathcal{M}}(a)]}{5\text{-term}}$

[de Jeu does not maybe prove rel of [illegible] logs?]

$\mathcal{H}_\bullet(F)$ - Hopf alg. (graded, commutative) $\quad \mathcal{L}_\bullet(F)$ - Lie coalgebra

$$\Delta \mathrm{Li}_2^{\mathcal{M}}(a) = 1 \otimes \mathrm{Li}_2^{\mathcal{M}}(a) + \mathrm{Li}_2^{\mathcal{M}}(a) \otimes 1 + \log^{\mathcal{M}}(a) \otimes \log^{\mathcal{M}}(1-a)$$

kills 5-term (can derive via cluster alg's)

In gen^l $\exists$ formula for $\Delta \mathrm{Li}^{\mathcal{M}}_{n_1,\dots,n_k}(a_1,\dots,a_k) = \dots$

($\rightarrow$ reduce fct'l eq's to lower order ones)

Meat of Gon Prog: can be described differently, via Hopf alg of MTMs

cohom of $\mathcal{H}$, or rather of $\mathcal{L}$ (Chevalley-Eilenberg)

$$\Big[\overset{i=1}{\mathcal{L}} \xrightarrow{\Delta} \overset{i=2}{\Lambda^2 \mathcal{L}} \to \Lambda^3 \mathcal{L} \to \dots \to \Lambda^n \mathcal{L}_1\Big]_n$$

for fixed weight set fin. ~~dims~~ length.

eg $[H^1(\mathcal{L})]_n = \ker(\Delta)$,

where $\Delta : \mathcal{L}_n \to \bigoplus_{\substack{i\leq j \\ i+j=n}} \mathcal{L}_i \wedge \mathcal{L}_j$

Conj (Goncharov, inspired by Zagier, and Deligne-Beilinson)

$$[H^i(\mathcal{L}(F))]_n \overset{?}{=} gr^{(n)} K_{2n-i}(F)_{\mathbb{Q}}$$

fundamental, hard to define + nice properties

$$= H^i_{\mathcal{M}}(\mathrm{Spec}(F), \mathbb{Q}(n))$$

Fact: $H^1(F, \mathbb{Q}(n)) \cong H^1(F(t), \mathbb{Q}(n))$ $\Rightarrow$ morally existence of fct'l eqs

Known in wt 2:

$$B_2(F) \to \Lambda^2 F^\times$$

What is a fundamental reason for this?

Periods: numbers arising from $\int_{alg} alg$.

EX: $\log(a) = \int_1^a \frac{dt}{t}$, $Li_2(a) = \int_{0<t_1<t_2<1} \frac{a\, dt_1 dt_2}{(1-at_1)t_2}$

General setting for such integral settings

X - sm proj. $/\mathbb{C}$

$\omega \in \Omega^{top}(X \setminus D_1)$, γ cycle on X with bdry on D_2

$\rightarrow \int_\gamma \omega$, so think of periods as triples

cohom. interpret: Betti coh. $H^n_B(X \setminus D_1, D_2 \setminus (D_1 \cap D_2))$ (F) $\mathbb{Q}$-spaces

alg dR coh. $H^n_{dR}($ " , " $)$

Isom Betti $\to$ dR $\otimes \mathbb{C}$

$\rightsquigarrow$ matrix ("period matrix")

For MPL's, see other MPL's in Δ.

Last (+ hardest) step: alg var's have extra structure

one bit of this is "MHS" as very sophisticated LA narrative (2 filtr.s with special prop) $\otimes$ multiply, take duals,

$\to$ above are MHS's! Can derive the coproduct Δ

MHS from alg geom $\to$ Motives [Universality conj: all MHS from MPL's subclass of MTM (from subseq exts of —) | Strong predictive power of Δ]

Goncharov's programme

Danil Rudenko

II

II The depth conjecture

Recall: Most hard part: MPL's $\longleftrightarrow$ MTMs

F field

$$\mathcal{L}_n(F) = \left\{ \mathbb{Q}\text{-span } \mathrm{Li}^{\mathcal{M}}_{n_1 \dots n_k}(a_1, \dots, a_k) \right\}_{a_i \in F} \Big/ \text{fct. eq's of polylogs}$$

think "univ gp of polylog"

"$\mathcal{L}$" $\leftarrow$ mod products ...

$$\mathcal{L}_1(F) = \left(\mathbb{Q}\, \mathrm{Li}^{\mathcal{M}}_1(a) \Big/ \mathrm{Li}_1(1-a) + \dots \right) = F^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$\mathcal{L}_2(F) = B_2(F) = \mathbb{Q}[F] \big/ \text{5-term}$$

(key)

Conjecture:

$$\mathrm{CE}: \left[\mathcal{L}_\bullet \xrightarrow{\Delta} \Lambda^2 \mathcal{L}_\bullet \to \Lambda^3 \mathcal{L} \to \dots \right]$$

$$H^i\left(\mathcal{L}_n \to \mathcal{L}_{n-1} \otimes \mathcal{L}_1 + \dots \to \quad \dots \to \Lambda^2 \mathcal{L}_1 \right)$$

$$= H^i_{\mathcal{M}}(F, \mathbb{Q}(n)) = \mathrm{gr}_n^{\gamma} K_{2n-i}(F)$$

F - number field $\rightsquigarrow$ fin dim K-gps

$$\mathrm{Ker}(\Delta) = H^1_{\mathcal{M}}(F, \mathbb{Q}(n))$$

$$\mathcal{L} \xrightarrow{\Delta} \Lambda^2\mathcal{L} \to \Lambda^3\mathcal{L} \to \Lambda^4\mathcal{L} \to \dots$$

$$\cup\!\!\shortmid$$

$$B_n \xrightarrow{\Delta} B_{n-1} \otimes F^\times \to B_{n-2} \otimes \Lambda^2 F^\times \to \dots \quad (*)$$

<u>Conj</u> q is? Why should the "tiny" space of class polylogs be sufficient?

Could this be an artefact of # flds?

$\mathcal{L}_n(F)$ $\qquad \Delta = \sum_{1 \le i \le j} \Delta_{ij}$

$\qquad \overline{\Delta} = \sum_{2 \le i \le j} \Delta_{ij}$: <u>kills</u> class polylogs?

$B_n \subset \ker(\overline{\Delta})$ $\qquad$ nat'l map $0 \to F^\times \to \mathcal{L} \to \mathcal{L}_{\geq 2} \to 0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Delta \qquad \overline{\Delta}$

<u>Freeness Conjecture:</u>

- $B_n = \ker(\overline{\Delta}|_{\mathcal{L}_n})$
- $(\mathcal{L}_{\geq 2}, \overline{\Delta})$ is cofree (no "co-relations"?) ($\Leftrightarrow$ cohom $= 0$ exc for H^1)

$\mathcal{L}_2 = B_2$ (in partic $Li_{1,1}$ can be expr. via Li_2)

$\mathcal{L}_3 = B_3$ ($\rightsquigarrow Li_{1,1,1}, Li_{2,1}, Li_{1,2} \to Li_3$)

now gets interesting

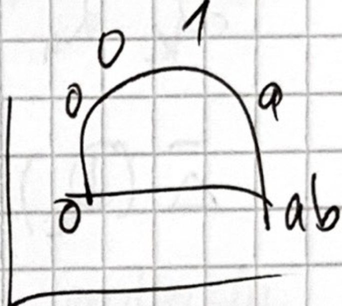
$$0 \to B_4 \to \mathcal{L}_4 \xrightarrow{\overline{\Delta}} \Lambda^2 B_2 \to 0$$

What does it mean explicitly

Can reduce fct space to $Li_{3,1}$ & Li_4

look for fct $? \to Li_2(a) \wedge Li_2(b)$

$Li_{2;1,1}(a,b)$



divergent, make sense via renormalis.

like MZS

proves "easy half"

kernel

- $Li_{2;1,1}(a,b) + Li_{2;1,1}(a,1-b) \in B_4$ (Zagier)

- $Li_{2;1,1}(a, \boxed{V}(b,c))$

Motivic philosophy does not give any clue.

[122 terms]

$$\mathcal{L}_5 \to 0 \to B_5 \to \mathcal{L}_5 \to B_2 \otimes B_3 \to 0$$

$$0 \to B_6 \to \mathcal{L}_6 \to \begin{matrix} \mathcal{L}_4 \otimes B_2 \\ \oplus \\ B_3 \wedge B_3 \end{matrix} \to \bigwedge^3 B_2 \to 0$$

$\mathcal{L}, \mathcal{L}_{3,2}$ spectral seq, Hochschild-Serre, cofree

only $H^1(\ ,\ )$

$\rightsquigarrow$ only part that remains is (*)

What about higher depth? (prettiest bit)

Take $\mathcal{L}$,

$D_k \mathcal{L}_n =$ polylogs of wt n depth $\leq k$

$\bar{\Delta}(D_k) \subseteq \sum_{\substack{i+j=k \\ 1 \leq i,j}} D_i \wedge D_j$

<u>Depth Conj</u> | so $(\mathrm{gr}^D \mathcal{L}, \bar{\Delta})$ Lie coalg
is cofree Lie coalgebra on $D_1 = B_2 \oplus B_3 \oplus \dots$

have map $D_2 / D_1 \xrightarrow{\bar{\Delta}} \Lambda^2 D_1$ in fact $\cong$!

Similarly $D_3 / D_2 \longrightarrow \dfrac{D_1 \otimes S^2 D_1}{S^3 D_1}$

---

wt 6, depth 3

<u>Conj</u> $D_3 \mathcal{L}_6 / D_2 \mathcal{L}_6 \overset{?}{\cong} \dfrac{B_2 \otimes S^2 B_2}{S^3 B_2}$

$\mathrm{Li}_{3,1,1,1}(a,b,c) \longrightarrow \mathrm{Li}_2(a) \otimes \mathrm{Li}_2(b) \otimes \mathrm{Li}_2(c)$ (only 1 term)

Try inverse map, get into same problems as HLG.

$\mathrm{Li}_{3,1,1,1}(a,b,c) + \mathrm{Li}_{3,1,1,1}(a,b,1-c) \in D_2$

" $(a,b,\bar{V}(c,d)) \in D_2$

What do we need in $\mathfrak{gl}^e$?

- ~~Fctl.~~ Fctl. eq's for $Li_{n_1,\dots,n_k}$
- Be able to prove Gang! formula + higher analogues w/o computations

Idea: Use Cluster algebras

- Construct bigrassmannian cocycle explicitly

How to use C.A.? "Tiny bit"

Space of cluster polylogs, can implement on computer

gives space, in any wt — only quadr. polylogs

~~Dept~~ 6 : Depth 2 is now known

Workshop Dinner

Fogo de Chao

UMD motorcoach

SESO lobby 6pm

Cluster polylogarithms

Daniil Rudenko

III

Clarify · $A = \mathbb{Q}$-v.s. cofree coLie alg.

$\Rightarrow \mathrm{CoLie}_n(A) = \dfrac{A^{\otimes n}}{ш}$

(For $\overset{A_n}{\mathbb{Q}}\langle a_1, \dots, a_n\rangle$ get Lyndon words ...)

$\mathrm{CoLie}_1(A) = A$

$\mathrm{CoLie}_2(A) = \Lambda^2 A$ $\quad (v_1, v_2]$

$\mathrm{CoLie}_3(A) = S^2 A \otimes A / S^3 A$

For Daniil cf ZeC!

⋮

Depth Conj: weight n $\quad$ quot is zero if $k > \frac{n}{2}+1$

$$D_k \mathcal{L} / D_{k+1}\mathcal{L} \xrightarrow{\sim} \mathrm{CoLie}_k(B_2 \otimes B_3 \otimes B_4 \otimes \dots)$$

Thm: (Rudenko) Any $\mathrm{Li}_{n_1,\dots,n_k}(a_1,\dots,a_k)$ of wt n can be expressed via polylogs of depth $\le n/2$

Cluster polylogs

~~Defn~~ Let X be an alg var, affine smooth

coll of 1-forms $\omega_1, \dots, \omega_n \in \Omega^1_X$, $d\omega_i = 0$

$\gamma : [0,1] \longrightarrow X(\mathbb{R})$ $\qquad \boxed{f_i(t_i) = \gamma^* \omega_i}$

$$\int_\gamma \omega_1 \circ \dots \circ \omega_n = \int_{0 \le t_1 \le \dots \le t_n \le 1} f_1(t_1) \dots f_n(t_n)$$

(Chen)

Fundamental: depends on path;

~~Restr~~ condition under which only dep on homot. class

(~~ibtg~~ integr. cond.)

$$\int_\gamma : \Omega_1^{\otimes n} \longrightarrow \mathbb{C}$$

$$(\omega_1 | \dots | \omega_n) \longrightarrow \int_\gamma \omega_1 \circ \dots \circ \omega_n$$

Integr. cond.

$$\int_\gamma \Big(\underbrace{\sum q_i}_{\in \mathbb{Q}} [\omega_1^i | \dots | \omega_n^i]\Big)$$ depends only on the homot class of γ

iff $\forall\, 1 \le s \le n-1$ $\sum_i q_i [\omega_1^i | \dots | \omega_{s-1}^i] \otimes \underbrace{(\omega_s \wedge \omega_{s+1})}_{\in \Omega^2} \otimes (\dots)$

$$\underbrace{\Omega^1 \otimes \dots \Omega^1}_{s} \otimes \Omega^2 \otimes \underbrace{\Omega^1 \otimes \dots \otimes \Omega^1}_{n-s-2}$$

Ex std iter int.

$$I(x_0; x_1 \longrightarrow x_n; x_{n+1}) = \int_{x_0}^{x_{n+1}} \frac{dx}{x - x_1} \circ \dots \circ \frac{dx}{x - x_n}$$

With this in mind (as motivation), define cluster polylogs

X = cluster variety $\begin{pmatrix} \text{eg } Gr(k,n) \\ M_{0,n} \\ \vdots \end{pmatrix}$

cluster coordinates

$A = \mathbb{Q}\text{-span } \frac{da}{a} \subseteq \mathcal{O}(X)^\times$

symbol spanned by cluster A coords

$\mathcal{L}_n(X) \subseteq \mathcal{L}ie_n(A)$

span of cluster polylogs of wt n

<u>n=1</u> $\mathcal{L}_1(X) = A$

<u>n=2</u> $\mathcal{L}_2(X) \subseteq \Lambda^2 A$

Meinrel (mut.)

$aa' = M_1 + M_2$

spaned by elts $\frac{M_1}{aa'} \wedge \frac{M_2}{aa'}$

$\equiv$ cllog (cluster X-coord)

$X \wedge (1+X)$

Cluster polylogs of wt n

$\mathcal{L}_n(X) : \sum n_i [a_{i1} | \ldots | a_{in}] \in A^{\otimes n}$

subj to two cond's

(1) Integr. cond (+ extra detail : require relations to come from $\mathcal{L}_2$

ie $\sum n_i (\quad] \otimes a_s^i \otimes a_{s+1}^i \otimes [\ldots]$

$\in \mathcal{L}_1 \otimes \ldots \otimes \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_1 \otimes \ldots \mathcal{L}_1$

key cond:

(2) $\forall i : \exists$ cluster containing $a_{i1} \ldots a_{in}$

project $\mathcal{L}_n(X)$

Ex: $M_{0,4} \longleftrightarrow Gr(2,4)$

X-coord $= -\frac{M_1}{M_2}$ $\quad Li_n^M\left(-\frac{M_1}{M_2}\right) \to \left[\frac{M_1}{M_2} \middle| \ldots \middle| \frac{M_1}{M_2} \middle| \frac{M_1}{aa'}\right]$

note a & a' are not in same cluster

Eqn: 7 pts $\rightsquigarrow$ can pull back

$$\sum(-1)^i\, QLi_4(x_1,\dots,\hat{x}_i,\dots,x_7) + \dots = 0$$

21 terms $\to$ $Li_{3,1}$, Li_4

$QLi_m(x_0,\dots,x_{2m+1}) = \sum Li_{\dots,1}((\;)(\;)(\;)(\;))$

decomp into 4-gons

Why unique?

$Li_{1,1}(a,b) \leftrightarrow 1-a,\ 1-b,\ 1-ab$ | branch locus where prod. of succ. vars $=1$

All "monsters" disappear in lin comb's

(n,N) s.t. $N \geq n+3$

$$\rightsquigarrow \sum_{i_0<i_1<\dots<i_{2r+1}\leq m+1} (-1)^{i_0+\dots+i_{2r+1}}\, QLi_n(x_{i_0},\dots,x_{i_{2r+1}}) = 0$$

Spec. cases: $N=5,\ m=2 \rightsquigarrow$ 5-term Li_2

$N=6,\ m=3 \rightsquigarrow QLi_3(x_1,\dots,x_6) \pm \dots = 0$

3 terms $\to Li_{2,1}(\;,\;) + \dots + \underset{(\text{several})}{Li_3} = 0$

Other spec cases

$QLi_{2n}(\;,\;x_{2n+1}) \in D_n$

$QLi_{2n+1}(\;,\;x_{2n+1}) \in D_n$

Mon cohom is Koszul (via descr. of Mon) $\rightsquigarrow$ each iter. int has depth $\leq 1/2$ wt

Different description

Thm (R. Matveiakin)

$$CL_n(\mathcal{M}_{0,m}) = \left\{ \begin{array}{l} \mathrm{span}(x_1, \dots, x_m) \\ I(x_{i_0}, x_{i_1}, \dots, x_{i_n}, x_{m+1}) \quad i_j \le i_{j+1} \end{array} \right\} \cap (\text{PGL-invar})$$

$\rightsquigarrow I(x_0, x_1, \dots, x_4, x_5) +$ degen's

"correct" lin comb via degenerate ones

$\longrightarrow$ PGL-invar

Finally, dim of space

$$\dim CL_n(\mathcal{M}_{0,m+2}) = \binom{m+1}{3} + \binom{m+1}{4} + \dots + \binom{m+1}{m+1}$$

$(m+1\ (?))$

Conj: for any finite type cluster alg

What happens on Grassmannian? (in gen^l ∞-dim)

$Gr(k,n)$ only Plücker clusters

				Li_3
$Gr(3,6)$	19	31	$=30+1$ 33	33
$Gr(3,7)$	55	111 $(=112-1)$	132 $(133-1)$	133

$Li_2([1|2345])$

"+1": $Li_{2,1}([1|2346]\,[2|1456])$ (?)

Alt (?)

(extra det ...)