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K_3 OF A FIELD AND THE BLOCH GROUP

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ABSTRACT. For an arbitrary field F the author constructs a short exact sequence connecting the indecomposable part of the group $K_3(F)$ with the Bloch group $\mathrm{B}(F)$ of F. Unique divisibility of $\mathrm{B}(F)$ for algebraically closed F is proved.

Bibliography: 18 titles.

One of the serious difficulties encountered in the higher K-theory of fields is the deviation between the K-theories of Milnor and Quillen. This difficulty first appears in dimension three. It is known [3] that the kernel of the canonical homomorphism $K_3^M(F) \to K_3(F)$ is annihilated by multiplication by two, and there are serious grounds for expecting that this kernel is always trivial. At the same time, the group $K_3(F)_{\mathrm{nd}} = \mathrm{Coker}(K_3^M(F) \to K_3(F))$ can be very large, as shown by examples of numerical fields. Bloch [5] determined a connection between $K_3(F)_{\mathrm{nd}}$ and the group describing the nontrivial relations on the tensors of the form $x \otimes (1-x)$ $(x \in F^*,\ x \neq 1)$. The latter group is now called the Bloch group of the field F and denoted by $\mathrm{B}(F)$. This group is encountered also in certain other problems of K-theory, for example, in the computation of the homology of $\mathrm{SL}_2(F)$ (see [6]). One of the basic results in the present article (Theorem 5.2) gives a precise relation between $K_3(F)_{\mathrm{nd}}$ and the Bloch group: for an arbitrary field F we have the exact sequence $0 \to \mathrm{Tor}(F^*, F^*)^\sim \to K_3(F)_{\mathrm{nd}} \to \mathrm{B}(F) \to 0$, where $\mathrm{Tor}(F^*, F^*)^\sim$ is the unique nontrivial extension of $\mathrm{Tor}(F^*, F^*)$ by $\mathbb{Z}/2$.

Most of the results in this paper were obtained in 1982. The purpose of the investigation was to prove the unique divisibility of $\mathrm{B}(F)$ for algebraically closed F, which is equivalent to the validity of the Quillen-Lichtenbaum conjecture for K_3. However, the method worked out here turned out to be applicable for proving the Quillen-Lichtenbaum conjecture in full scope ([17], [7], [18]), which diminished the interest in studying $\mathrm{B}(F)$, and the results were not published. During the intervening time a number of papers have appeared containing results close to ours (see especially [13]) but not covering them. These papers demonstrate interest in this circle of questions, and that prompted the author to publish his investigation.

Some words on the notation. The symbol Σ_n denotes the group of permutations of n elements, which is identified with the corresponding subgroup of $\mathrm{GL}_n(F)$, $\mathrm{GM}_n(F)$ (respectively, B_n, T_n) denotes the subgroup of monomial (respectively, upper triangular, diagonal) matrices in $\mathrm{GL}_n(F)$. Whenever the coefficients are not indicated in the consideration of the homology of a group,

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we have in mind integral homology. If there is a homomorphism $A \to B$ of Abelian groups, we sometimes write B/A instead of $B/\operatorname{Im} A$ and take other liberties of this kind.

All fields considered in this article are assumed to be infinite.

§1. The Bloch group

For an arbitrary field F denote by $\mathscr{D}(F)$ the free Abelian group with basis $[x]$, where $x \in F^* - 1$. Consider next the homomorphism $\varphi: \mathscr{D}(F) \to F^* \otimes F^*$, given by the formula $\varphi([x]) = x \otimes (1-x)$. The following lemma can be proved by uncomplicated computations.

LEMMA 1.1. *If* $x \neq y \in F^* - 1$, *then*

$$\varphi\left([x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right]\right) = x \otimes \left(\frac{1-x}{1-y}\right) + \left(\frac{1-x}{1-y}\right) \otimes x. \tag{1.1}$$

Denote by σ the involution of $F^* \otimes F^*$ given by the formula $\sigma(x \otimes y) = -(y \otimes x)$, and by $(F^* \otimes F^*)_\sigma$ the quotient of $F^* \otimes F^*$ with respect to the action of σ. Moreover, denote by $\mathfrak{p}(F)$ the factor group of $\mathscr{D}(F)$ by the subgroup generated by the elements of the form $[x]-[y]+[y/x]-[(1-x^{-1})/(1-y^{-1})]+[(1-x)/(1-y)]$. According to Lemma 1.1, the homomorphism φ induces a new homomorphism $\mathfrak{p}(F) \to (F^* \otimes F^*)_\sigma$, which we denote by the same letter φ. The kernel of the latter homomorphism is denoted by $\mathrm{B}(F)$ and called the Bloch group of the field F. Thus, we get the exact sequence

$$0 \to \mathrm{B}(F) \to \mathfrak{p}(F) \to (F^* \otimes F^*)_\sigma \to \mathrm{K}_2(F) \to 0. \tag{1.2}$$

For $x \in F^* - 1$ we set $\langle x \rangle = [x] + [x^{-1}]$; moreover, we set $\langle 1 \rangle = 0$.

LEMMA 1.2. $x \mapsto \langle x \rangle$ *is a homomorphism from* F^* *to the subgroup of elements of exponent* 2 *in* $\mathfrak{p}(F)$.

PROOF. Let $x \neq y \in F^* - 1$. In the group $\mathfrak{p}(F)$ we have the equalities

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)] = 0,$$
$$[x^{-1}] - [y^{-1}] + [x/y] - [(1-x)/(1-y)] + [(1-x^{-1})/(1-y^{-1})] = 0.$$

Adding these equalities, we get that $\langle x \rangle - \langle y \rangle + \langle y/x \rangle = 0$. Interchanging the roles of x and y and taking into account that $\langle y/x \rangle = \langle x/y \rangle$, we see that $2 \cdot \langle y/x \rangle = 0$. Since any element $z \in F^* - 1$ can be represented in the form y/x (where $x \neq y \in F^* - 1$), we conclude that $2 \cdot \langle z \rangle = 0$ for any $z \in F^*$. We have already verified the relation $\langle x \rangle + \langle y/x \rangle = \langle y \rangle$ under the condition $x \neq y \in F^* - 1$. If $x = 1$ or $x = y$, then this relation is trivial. If $y = 1$, then it takes the form $\langle x \rangle + \langle x^{-1} \rangle = 0$. This relation is valid, because $\langle x^{-1} \rangle = \langle x \rangle$ and $2 \cdot \langle x \rangle = 0$.

LEMMA 1.3. *The relation* $[x] + [1-x] = [y] + [1-y]$ *holds for any elements* $x, y \in F^* - 1$.

PROOF. It can be assumed that $x \neq y$, and then the following relations hold in the group $\mathfrak{p}(F)$:

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)] = 0,$$
$$[1-y] - [1-x] + [(1-x)/(1-y)] - [(1-x^{-1})/(1-y^{-1})] + [y/x] = 0.$$

Subtracting one from the other, we get the desired equality.

The element $[x]+[1-x] \in \mathrm{B}(F)$, which, as just shown, is independent of the choice of $x \in F^* - 1$, will be denoted by c (or c_F).

LEMMA 1.4. $3c = \langle -1 \rangle$.

PROOF. If $x \in F^* - 1$, then $3c = [x] + [1-x] + [x^{-1}] + [1-x^{-1}] + [(1-x)^{-1}] + [1-(1-x)^{-1}] = \langle x \rangle + \langle 1-x \rangle + \langle 1-x^{-1} \rangle = \langle -(1-x)^2 \rangle = \langle -1 \rangle$.

LEMMA 1.5. a) $6c = 0$; b) *if the equation* $x^2+1=0$ *is solvable in* F, *then* $3c = 0$; c) *if the equation* $x^2 - x + 1 = 0$ *is solvable in* F, *then* $2c = 0$; d) *if the equations* $x^2+1=0$, $x^2-x+1=0$ *are solvable in* F, *then* $c = 0$.

PROOF. The parts a) and b) follow from 1.2 and 1.4. Let $x^2 - x + 1 = 0$; then $x^3 = -1$ and $c = [x] + [1-x] = [x] + [x^{-1}] = \langle x \rangle = \langle -x^4 \rangle = \langle -1 \rangle = 3c$, i.e., $2c = 0$.

We now show, following Sah [13], that in $\mathrm{B}(\mathbb{R})$ the element c has order exactly six. To do this we consider the group $\mathfrak{p}'(\mathbb{R})$ given by the generators $[x]'$ $(0 < x < 1)$ and the relations

$$[x]' - [y]' + \left[\frac{y}{x}\right]' - \left[\frac{1-x^{-1}}{1-y^{-1}}\right]' + \left[\frac{1-x}{1-y}\right]' = 0 \qquad (0 < y < x < 1).$$

There is an obvious homomorphism $p\colon \mathfrak{p}'(\mathbb{R}) \to \mathfrak{p}(\mathbb{R})$. As is clear from the proof of Lemma 1.3, the element $c' = [x]' + [1-x]' \in \mathfrak{p}'(\mathbb{R})$ does not depend on the choice of x $(0 < x < 1)$ and passes into c under the homomorphism under consideration. In particular, $6c' \in \operatorname{Ker} p$.

LEMMA 1.6. *The homomorphism* p *is surjective, and its kernel coincides with the cyclic subgroup generated by* $6c'$.

For the proof we construct the inverse homomorphism $\mathfrak{p}(\mathbb{R}) \to \mathfrak{p}'(\mathbb{R})/6c'$. To do this we first define a homomorphism $\psi\colon \mathscr{D}(\mathbb{R}) \to \mathfrak{p}'(\mathbb{R})/6c'$ by the formula

$$\psi([x]) = \begin{cases} [x]' & \text{for } 0 < x < 1, \\ -[x^{-1}]' & \text{for } 1 < x, \\ c' + [(1-x)^{-1}]' & \text{for } x < 0. \end{cases}$$

The element $[x]-[y]+[y/x]-[(1-x^{-1})/(1-y^{-1})]+[(1-x)/(1-y)] \in \mathscr{D}(\mathbb{R})$ will be denoted by $[x, y]$. We must show that $\psi([x, y]) = 0$ for $x \neq y \in \mathbb{R}^* - 1$. For this we first observe that, by construction, $\psi([x] + [1-x]) = c'$ for any $x \in \mathbb{R}^* - 1$, and

$$\psi([x] + [x^{-1}]) = \begin{cases} 0 & \text{for } x > 0, \\ 3c' & \text{for } x < 0. \end{cases}$$

Now, since

$$[x, y] - [1-y, 1-x] = ([x] + [1-x]) - ([y] + [1-y]),$$

$$[x, y] + [x^{-1}, y^{-1}] = ([x] + [x^{-1}]) - ([y] + [y^{-1}]) + ([y/x] + [x/y]),$$

$$[x, y] + [y, x] = \left(\left[\frac{y}{x}\right] + \left[\frac{x}{y}\right]\right) - \left(\left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-y^{-1}}{1-x^{-1}}\right]\right) + \left(\left[\frac{1-x}{1-y}\right] + \left[\frac{1-y}{1-x}\right]\right)$$

and the right-hand sides of these relations are carried into zero under the action of ψ, it follows that in verifying the formula $\psi([x, y]) = 0$ we may replace $[x, y]$ by $[1-y, 1-x]$, $[x^{-1}, y^{-1}]$, $[y, x]$. This enables us to reduce the general case to the case $0 < y < 1$, $y < x$. If $x < 1$, then everything is obvious. If $x > 1$, then

$$\begin{aligned}\psi([x, y]) &= -[x^{-1}]' - [y]' + \left[\frac{y}{x}\right]' - \left(c' + \left[\left(1 - \frac{1-x^{-1}}{1-y^{-1}}\right)^{-1}\right]'\right) \\ &\quad + \left(c' + \left[\left(1 - \frac{1-x}{1-y}\right)^{-1}\right]'\right) \\ &= -[x^{-1}]' - [y]' + \left[\frac{y}{x}\right]' - \left[\frac{x(1-y)}{x-y}\right]' + \left[\frac{1-y}{x-y}\right]'.\end{aligned}$$

Finally, note that $0 < y/x < y < 1$, hence the equality

$$[y]' - \left[\frac{y}{x}\right]' + [x^{-1}]' - \left[\frac{1-y}{1-x}\right]' + \left[\frac{x(1-y)}{x-y}\right]' = 0$$

is valid in $\mathfrak{p}'(\mathbb{R})$.

Below we need the so-called Rogers L-function. Recall that this function is analytic on the open interval $(0, 1)$, where it is given by the formula $L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2}\ln(x)\ln(1-x)$. We extend L by continuity to the closed interval $[0, 1]$, with $L(0) = 0$ and $L(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Note also the following formula for the derivative:

$$L'(x) = -\frac{1}{2}\left[\frac{\ln(1-x)}{x} + \frac{\ln(x)}{1-x}\right].$$

Let $0 < y < x < 1$. Differentiating the expression

$$L(x) - L(y) + L\left(\frac{y}{x}\right) - L\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + L\left(\frac{1-x}{1-y}\right)$$

with respect to x and using the formula for L', we see without difficulty that this expression does not depend on x. Letting x go to 1, we get that

$$L(x) - L(y) + L\left(\frac{y}{x}\right) - L\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + L\left(\frac{1-x}{1-y}\right) = L(1) = \frac{\pi^2}{6}.$$

Thus, the correspondence $[x]' \to L(x) - L(1)$ determines a homomorphism $\mathfrak{p}'(\mathbb{R}) \to \mathbb{R}$, and the image of c' under this homomorphism is

$$L(0) + L(1) - 2L(1) = -\frac{\pi^2}{6} \neq 0.$$

This shows that $c' \in \mathfrak{p}'(\mathbb{R})$ is an element of infinite order. Finally, we get the following proposition.

PROPOSITION 1.1. *If the field F is formally real, then the order of the element $c_F \in \mathrm{B}(F)$ is equal to six.*

§2. The Bloch group and $H_3(\mathrm{GL}_2(F))$

Let us consider the complex $C_* = C_*(F)$ in which C_p is the free Abelian group with basis $(x_0, \ldots, x_p)$, where the x_i are distinct points of $\mathbb{P}^1(F)$, and the differential is given by the formula

$$d(x_0, \ldots, x_p) = \sum_{i=0}^{p}(-1)^i(x_0, \ldots, \hat{x}_i, \ldots, x_p).$$

There is a natural augmentation $\varepsilon\colon C_0(F) \to \mathbb{Z} : \varepsilon(x_0) = 1$.

LEMMA 2.1. *The augmented complex* $C_* \to \mathbb{Z} \to 0$ *is acyclic.*

PROOF. Let $z = \sum n_i(x_0^i, \ldots, x_p^i)$ be a p-dimensional cycle of the augmented complex $(p \geq -1)$. Since F is infinite, we can find a point $x \in \mathbb{P}^1(F)$ different from all the x_j^i. It is now easy to see that $z = d(\sum n_i(x, x_0^i, \ldots, x_p^i))$.

The natural action of the group $G = \mathrm{GL}_2(F)$ on $\mathbb{P}^1(F)$ determines an action of G on C_* and turns C_* into an acyclic G-complex over $\mathbb{Z}$. According to the theorem on comparison of resolutions, there exists a homomorphism $C_*(G) \to C_*$ of G-complexes over $\mathbb{Z}$, unique up to homotopy, where $C_*(G)$ is the G-free resolution of $\mathbb{Z}$. This homomorphism induces canonical mappings $H_i(G) = H_i(C_*(G)_G) \to H_i((C_*)_G)$.

LEMMA 2.2. $H_3((C_*)_G) = \mathfrak{p}(F)$.

PROOF. The action of G on the bases of C_0, C_1, and C_2 is transitive, and hence $(C_p)_G = \mathbb{Z}$ for $p = 0, 1$, and 2. Every orbit of the action of G on the basis of C_p for $p \geq 3$ contains a unique element of the form $(0, \infty, 1, x_1, \ldots, x_{p-2})$, where the $x_i \in F^* - 1$ are distinct. Denoting the corresponding orbit by $[x_1, \ldots, x_{p-2}]$, we see that

$$(C_p)_G = \prod_{\substack{x_j \in F^* - 1, \\ x_i \neq x_j \text{ for } i \neq j}} \mathbb{Z}[x_1, \ldots, x_{p-2}].$$

Finally, it is easy to see that $d\colon (C_p)_G \to (C_{p-1})_G$ is given by the formula

$$d([x_1, \ldots, x_{p-2}]) = \left[\frac{1-x_1}{1-x_2}, \ldots, \frac{1-x_1}{1-x_p}\right] - \left[\frac{1-x_1^{-1}}{1-x_2^{-1}}, \ldots, \frac{1-x_1^{-1}}{1-x_p^{-1}}\right] + \left[\frac{x_2}{x_1}, \ldots, \frac{x_p}{x_1}\right] + \sum_{i=1}^{p-2}(-1)^i[x_1, \ldots, \hat{x}_i, \ldots, x_{p-2}].$$

According to the foregoing, we get a canonical homomorphism $\partial\colon H_3 \times (\mathrm{GL}_2(F)) \to \mathfrak{p}(F)$.

THEOREM 2.1. ∂ *determines an isomorphism* $H_3(\mathrm{GL}_2(F))/H_3(\mathrm{GM}_2(F)) \xrightarrow{\sim} \mathrm{B}(F)$.

PROOF. Let us consider the hyperhomology spectral sequence $E_{pq}^1 = H_p(G, C_q) \Rightarrow H_{p+q}(G, \mathbb{Z})$. The action of G on the basis elements of C_0, C_1, and C_2 is transitive, and

$$\mathrm{Stab}(0) = B_2, \qquad \mathrm{Stab}(0, \infty) = T_2 = F^* \oplus F^*, \qquad \mathrm{Stab}(0, \infty, 1) = F^*.$$

Considering that $H_*(T_2) = H_*(B_2)$ ([3, §1]), we conclude that

$$E^1_{*1} = E^1_{*2} = H_*(F^* \oplus F^*), \qquad E^1_{*2} = H_*(F^*).$$

LEMMA 2.3. *The differential* $d^1: H_*(F^* \oplus F^*) = E^1_{*1} \to E^1_{*0} = H_*(F^* \oplus F^*)$ *coincides with* $\sigma - 1$, *where* σ *is the automorphism of* $H_*(F^* \oplus F^*)$ *induced by transposition of terms, and the differential*

$$d^2: H_*(F^*) = E^1_{*2} \to E^1_{*1} = H^*(F^* \oplus F^*)$$

coincides with Δ_*, *where* $\Delta: F^* \to F^* \oplus F^*$ *is the diagonal imbedding.*

PROOF. The isomorphisms $H_*(T_2, \mathbb{Z}) \xrightarrow{\sim} H_*(\mathrm{GL}_2(F), C_0)$ and $H_*(T_2, \mathbb{Z}) \xrightarrow{\sim} H_*(\mathrm{GL}_2(F), C_1)$ are induced by the following morphisms in the category of pairs (group, module):

$$\varphi_0: (T_2, \mathbb{Z}) \to (\mathrm{GL}_2(F), C_0) : (t, n) \mapsto (t, n \cdot (0)),$$
$$\varphi_{0,\infty}: (T_2, \mathbb{Z}) \to (\mathrm{GL}_2(F), C_1) : (t, n) \mapsto (t, n \cdot (0, \infty)).$$

In turn, d^1 is the difference between two homomorphisms, of which the first is induced by the homomorphism $C_1 \to C_0 : (x_0, x_1) \to (x_1)$ and the second by the homomorphism $C_1 \to C_0 : (x_0, x_1) \to (x_0)$. Consequently, the composition $H_*(T_2, \mathbb{Z}) \xrightarrow{\sim} H_*(\mathrm{GL}_2(F), C_1) \to H_*(\mathrm{GL}_2(F), C_0)$ is the difference between two homomorphisms, of which the first is induced by the morphism $\varphi_\infty: (T_2, \mathbb{Z}) \to (\mathrm{GL}_2(F), C_0) : (t, n) \mapsto (t, n \cdot (\infty))$ and the second by the morphism φ_0. Denote by s the permutation matrix $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. As is known, inner automorphisms act trivially on homology, and hence $(\varphi_\infty)_* = s_*(\varphi_\infty)_* = (\varphi_0\tau)_* = (\varphi_0)_*\tau_*$, where τ is transposition of terms of T_2. Finally, we get that the composition $(\varphi_0)_*^{-1} d^1 (\varphi_{0,\infty})_*$ coincides with $\tau_* - 1$. The second assertion is proved analogously.

Since Δ has a left inverse, Δ_* is a monomorphism, and thus $E^2_{*2} = 0$. Further, $H_1(F^* \oplus F^*) = F^* \oplus F^*$, and σ transposes terms. Moreover,

$$H_2(F^* \oplus F^*) = H_2(F^*) \oplus H_2(F^*) \oplus (F^* \otimes F^*),$$

σ transposes the copies of $H_2(F^*)$, and σ coincides on $F^* \otimes F^*$ with the involution denoted by the same letter in the preceding section. It follows from these remarks that the terms of interest to us in the spectral sequence have the following form:

$$\begin{array}{cccc} H_3(T_2)_\sigma & & & \\ H_2(F^*) \oplus (F^* \otimes F^*)_\sigma & (F^* \otimes F^*)^\sigma & & \\ F^* & 0 & 0 & \\ \mathbb{Z} & 0 & 0 & \mathfrak{p}(F) \end{array}$$

There is only one nontrivial differential starting from $E^*_{0,3}$, namely, the differential $d^3: \mathfrak{p}(F) \to H_2(F^*) \oplus (F^* \otimes F^*)_\sigma = \Lambda^2(F^*) \oplus (F^* \otimes F^*)_\sigma$.

LEMMA 2.4. $d^2([x]) = x \wedge (1 - x) - x \otimes (1 - x)$.

The proof is direct but a very long and tedious calculation, and we omit it.

COROLLARY 2.1. $E^{\infty}_{0,3} = \mathrm{B}(F)$.

The spectral sequence under consideration determines a filtration on the group $H_3(\mathrm{GL}_2(F))$, with $H_3(\mathrm{GL}_2(F))^{(0)} = \mathrm{Im}(H_3(T_2))$, there is a canonical epimorphism $H_2(T_2)^{\sigma} \twoheadrightarrow H_2(T_2)^{\sigma}/\Delta_*(H_2(F^*)) = (F^* \otimes F^*)^{\sigma} \twoheadrightarrow H_3(\mathrm{GL}_2(F))^{(1)/(0)}$, $H_3(\mathrm{GL}_2(F))^{(2)} = H_3(\mathrm{GL}_2(F))^{(1)}$, and the boundary effect (which obviously coincides with the homomorphism ∂ considered above) induces an isomorphism $H_3(\mathrm{GL}_2(F))/H_3(\mathrm{GL}_2(F))^{(2)} \xrightarrow{\sim} \mathrm{B}(F)$. Thus, it remains to show that $H_3(\mathrm{GL}_2(F))^{(2)} = \mathrm{Im}(H_3(\mathrm{GM}_2(F)))$. To do this we need an explicit description of the homomorphism $H_2(T_2)^{\sigma} \to H_3(\mathrm{GL}_2(F))^{(1)/(0)} \hookrightarrow H_3(\mathrm{GL}_2(F))/H_3(T_2)$.

For an arbitrary group G let $C_*(G)$ denote the standard G-free resolution of the trivial G-module $\mathbb{Z}$ so that $C_i(G)$ is the free left G-module $\mathbb{Z}[G^{i+1}]$ (with basis the elements $\langle g_1, \dots, g_i\rangle = (1, g_1, g_1g_2, \dots, g_1g_2\cdots g_i)$). We use the usual device to turn the $C_i(G)$ into right G-modules. For any left G-module M the homology groups $H_i(G, M)$ coincide with the homology of the complex $C_*(G) \otimes_G M$. Any element $g \in G$ determines an automorphism of the complex $C_*(G) \otimes_G M t_g\colon \langle g_1, \dots, g_i\rangle \otimes m \to \langle {}^g g_1, \dots, {}^g g_i\rangle \otimes gm$. This automorphism is homotopic to the identity, with the corresponding homotopy given by the formula

$$\rho_g(\langle g_1, \dots, g_i\rangle \otimes m) = \sum_{j=0}^{i} (-1)^j \langle g_1, \dots, g_j, g^{-1}, {}^g g_{j+1}, \dots, {}^g g_i\rangle \otimes m.$$

LEMMA 2.5. *Suppose that $u \in H_2(T_2)^{\sigma}$, h is a representing cycle for u; let $\tau(h) - h = db$, where τ is the automorphism of transposition of terms, and b is some three-dimensional chain of the group T_2. Then the image of u in $H_3(\mathrm{GL}_2(F))/H_3(T_2)$ coincides with the homology class of the three-dimensional cycle $b - \rho_s(h)$, where $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.*

PROOF. The spectral sequence under consideration coincides with the spectral sequence of the bicomplex $C_*(G) \otimes_G C_*(F)$, where $G = \mathrm{GL}_2(F)$. The augmentation $\varepsilon\colon C_*(F) \to \mathbb{Z}$ induces a homology equivalence $C_*(G) \otimes_G C_*(F) \xrightarrow{\varepsilon} C_*(G)_G$. The necessary computations are now collected in the diagram

$$\begin{array}{ccc} b - \rho_s(h) & \xleftarrow{\varepsilon} & b \otimes (0) - \rho_s(h) \otimes (\infty) \\ & & \big\downarrow d_G \\ h \otimes ((\infty) - (0)) & \xleftarrow{d_F} & h \otimes (0, \infty) \end{array}$$

Finally, we consider the Hochschild-Serre spectral sequence corresponding to the extension of groups $1 \to T_2 \to \mathrm{GM}_2(F) \to \Sigma_2 \to 1$, where $\Sigma_2 = \{1, s\}$ is the group of permutation matrices: $H_p(\Sigma_2, H_q(T_2)) \Rightarrow H_{p+q}(\mathrm{GM}_2(F))$. This spectral sequence determines a filtration on $H_3(\mathrm{GM}_2(F))$, with $H_3(\mathrm{GM}_2(F))^{(0)} = \mathrm{Im}(H_3(T_2))$; there is a canonical epimorphism $H_2(T_2)^{\sigma} \twoheadrightarrow H_1(\Sigma_2, H_2(T_2)) = H_2(T_2)^{\sigma}/(1+\sigma)(H_2(T_2)) \twoheadrightarrow H_3(\mathrm{GM}_2(F))^{(1)/(0)}$; $H_3(\mathrm{GM}_2(F))^{(2)/(1)} = 0$ (since $H_*(\Sigma_2, T_2) = 0$); $H_3(\mathrm{GM}_2(F)) = H_3(\mathrm{GM}_2(F))^{(2)} + H_3(\Sigma_2)$. By the foregoing, the homomorphism $H_3(\mathrm{GM}_2(F)) \to H_3(\mathrm{GL}_2(F))$ maps $H_3(\mathrm{GM}_2(F))^{(0)}$ onto $H_3(\mathrm{GL}_2(F))^{(0)}$. Further, it is easy to see that the image of a $u \in H_2(T_2)^{\sigma}$ in $H_3(\mathrm{GM}_2(F))/H_3(T_2)$ is given by the formula in Lemma 2.5. Consequently,

$H_3(\mathrm{GM}_2(F))^{(1)}$ is mapped onto $H_3(\mathrm{GL}_2(F))^{(1)}$. Thus, $\mathrm{Im}(H_3(\mathrm{GM}_2(F))) = \mathrm{Im}(H_3(\Sigma_2)) + H_3(\mathrm{GL}_2(F))^{(2)}$. Finally, the matrix s is similar to the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right) \in B_2$, and hence $\mathrm{Im}(H_3(\Sigma_2)) \subset \mathrm{Im}(H_3(B_2)) = \mathrm{Im}(H_3(T_2)) = H_3(\mathrm{GL}_2(F))^{(0)}$.

REMARK 2.1. The homology group $H_1(\Sigma_2, H_2(T_2))$ encountered in the computation of $H_3(\mathrm{GM}_2(F))$ is not hard to compute: it is equal to

$$(F^* \otimes F^*)^\sigma/(1+\sigma)(F^* \otimes F^*) = (\mu(F) \otimes \mu(F))^\sigma/(1+\sigma)(\mu(F) \otimes \mu(F))$$
$$= {}_2(\mu(F) \otimes \mu(F)) = \begin{cases} 0 & \text{if } F \supset \mu_{2\infty}, \\ \mathbb{Z}/2 & \text{otherwise.} \end{cases}$$

Thus, $\mathrm{Im}(H_3(\mathrm{GM}_2(F)) \to H_3(\mathrm{GL}_2(F))) = \mathrm{Im}(H_3(T_2))$ if F contains all the 2-primary roots of unity. However, if the group $\mu_{2\infty}(F)$ is finite and nontrivial, then it can be shown that $\mathrm{Im}(H_3(\mathrm{GM}_2(F)))$ is strictly larger than $\mathrm{Im}(H_3(T_2))$.

REMARK 2.2. For any n denote by $H_3^0(\mathrm{GM}_n(F))$ the kernel of the projection $H_3(\mathrm{GM}_n(F)) \to H_3(\Sigma_n)$. There is a canonical decomposition

$$H_3(\mathrm{GM}_n(F)) = H_3^0(\mathrm{GM}_n(F)) \oplus H_3(\Sigma_n).$$

As mentioned above, the image of $H_3(\Sigma_2)$ in $H_3(\mathrm{GL}_2(F))$ is contained in the image of $H_3(T_2) \subset H_3^0(\mathrm{GM}_2(F))$. Consequently, $\mathrm{Ker}\,\partial = \mathrm{Im}(H_3(\mathrm{GM}_2(F))) = \mathrm{Im}(H_3^0(\mathrm{GM}_2(F)))$.

REMARK 2.3. The theorem proved is a variant of the theorem of Bloch [6], and the method of proof also follows [6] on the whole. The form given for the theorem has the advantage that it is suitable for any (infinite) field.

§3. Construction of the homomorphism $\partial: H_3(\mathrm{GL}_3(F)) \to \mathrm{B}(F)$

Denote by $C_p^2(F)$ the free Abelian group with basis $(x_0, \dots, x_p)$, where the $x_i \in \mathbb{P}^2(F)$ are in general position. We define the differentials $d, d': C_p^2 \to C_{p-1}^2$ by the formulas

$$d(x_0, \dots, x_p) = \sum_{i=0}^{p} (-1)^i (x_0, \dots, \hat{x}_i, \dots, x_p),$$
$$d'(x_0, \dots, x_p) = \sum_{i=1}^{p-1} (-1)^k (x_0, \dots, \hat{x}_i, \dots, x_p).$$

The following assertion is proved just like Lemma 2.1.

LEMMA 3.1. *The following complexes are acyclic:*

$$0 \leftarrow \mathbb{Z} \xleftarrow{\varepsilon} C_0^2 \xleftarrow{d} C_1^2 \xleftarrow{d} \cdots,$$
$$0 \leftarrow C_0^2 \xleftarrow{d'} C_1^2 \xleftarrow{d'} \cdots$$

We define the homomorphism $\lambda: C_p^2 \to C_p^2$ by the formula $\lambda(x_0, \dots, x_p) = \sum_{k=0}^{p} (-1)^{pk} (x_k, x_{k+1}, \dots, x_{k+p})$ (where all the indices are reduced modulo $p+1$). An uncomplicated direct check proves the following fact.

LEMMA 3.2. $d'\lambda = \lambda d$.

Let us consider the following homomorphism of complexes:

$$\begin{array}{ccccccc} C_1^2 & \xleftarrow{d} & C_2^2 & \xleftarrow{d} & C_3^2 & \xleftarrow{d} & \cdots, \\ \lambda\downarrow & & \lambda\downarrow & & \lambda\downarrow & & \\ C_1^2 & \xleftarrow{d'} & C_2^2 & \xleftarrow{d'} & C_3^2 & \xleftarrow{d'} & \cdots \end{array}$$

and denote by $\mathscr{D}_*$ its cone; thus, $\mathscr{D}_0 = C_1^2$, $\mathscr{D}_p = C_p^2 \oplus C_{p+1}^2$ $(p \geq 1)$, and the differential is given by the matrix $\left(\begin{smallmatrix} -d & 0 \\ \lambda & d' \end{smallmatrix}\right)$. We define the augmentation $\varepsilon_{\mathscr{D}} : \mathscr{D}_0 \to \mathbb{Z}$ as the composition $\mathscr{D}_0 = C_1^2 \xrightarrow{d} C_0^2 \xrightarrow{\varepsilon} \mathbb{Z}$. It follows directly from Lemma 3.1 that the augmented complex $\mathscr{D}_* \to \mathbb{Z} \to 0$ is acyclic. There is a natural action of the group $GL_2(F)$ on $\mathscr{D}_*$ determining, as in §2, a canonical homomorphism $H_3(GL_3(F)) \to H_3((\mathscr{D}_*)_{GL_3(F)})$. The groups $(C_p^2)_{GL_3(F)} = \overline{C}_p^2$ are computed just as in §2. In particular, $\overline{C}_p^2 = \mathbb{Z}$ for $p \leq 3$, $\overline{C}_4^2 = \coprod \mathbb{Z} \cdot \left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]$, where $\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]$ is the orbit of

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & a & 0 \\ 0 & 0 & 1 & x & 1 \end{pmatrix}.$$

and the summation is over all a and x for which the indicated points are in general position, i.e., $a \neq x \in F^* - 1$; analogously, $\overline{C}_5^2$ is the free Abelian group with basis $\left(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix}\right)$ = the orbit of

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & a & b & 0 \\ 0 & 0 & 1 & x & y & 1 \end{pmatrix}$$

and the summation is over all $a, b, x, y \in F^* - 1$ such that $a \neq x$, $b \neq y$, $a \neq b, x \neq y$, $ay - bx \neq 0$.

Thus, the initial terms of the complex $(\mathscr{D}_*)_{GL_3(F)}$ have the form

$$0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\left(\begin{smallmatrix} -1 & 0 \\ 3 & 1 \end{smallmatrix}\right)} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{0} \mathbb{Z} \oplus \coprod \mathbb{Z} \cdot \begin{bmatrix} a \\ x \end{bmatrix}$$
$$\xleftarrow{\left(\begin{smallmatrix} -\bar{d} & 0 \\ \bar{\lambda} & \bar{d}' \end{smallmatrix}\right)} \coprod \mathbb{Z} \cdot \begin{bmatrix} a \\ x \end{bmatrix} \oplus \coprod \mathbb{Z} \cdot \begin{bmatrix} a & b \\ x & y \end{bmatrix},$$

and hence

$$H_3((\mathscr{D}_*)_{GL_3(F)}) = \operatorname{Coker}\left(\coprod \mathbb{Z} \cdot \begin{bmatrix} a \\ x \end{bmatrix} \oplus \coprod \mathbb{Z} \cdot \begin{bmatrix} a & b \\ x & y \end{bmatrix} \xrightarrow{\left(\begin{smallmatrix} -\bar{d} & 0 \\ \bar{\lambda} & \bar{d}' \end{smallmatrix}\right)} \mathbb{Z} \oplus \coprod \mathbb{Z} \cdot \begin{bmatrix} a \\ x \end{bmatrix} \right).$$

LEMMA 3.3. *Setting* $\rho(1) = 2c_F$, $\rho(\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]) = [a]$, *we get a well-defined homomorphism* $\rho: H_3((\mathscr{D}_*)_{\mathrm{GL}_3(F)}) \to \mathfrak{p}(F)$.

PROOF. We must verify that $\rho\left(\begin{smallmatrix} -\bar{d} & 0 \\ \bar{\lambda} & d' \end{smallmatrix}\right) = 0$. We carry out the necessary computations:

$$d'\left(\begin{bmatrix} a & b \\ x & y \end{bmatrix}\right) = \begin{bmatrix} a \\ x \end{bmatrix} - \begin{bmatrix} b \\ y \end{bmatrix} + \begin{bmatrix} b/a \\ y/x \end{bmatrix} - \begin{bmatrix} (1-a^{-1})/(1-b^{-1}) \\ \frac{y-b}{x-a} \cdot \frac{1-a}{1-b} \end{bmatrix} + \begin{bmatrix} (1-a)/(1-b) \\ \frac{1-a}{1-b} \cdot \frac{1-y}{1-x} \end{bmatrix}.$$

Applying ρ to the right-hand side, we clearly get zero. Further,

$$\begin{aligned} \rho\bar{\lambda}\left(\begin{bmatrix} a \\ x \end{bmatrix}\right) &= [a] + \left[\frac{x-a}{x}\right] + [x^{-1}] + \left[\frac{1-x}{a-x}\right] + \left[\frac{(1-a)x}{x-a}\right] \\ &= 2c_F + [a] - \left[\frac{a}{x}\right] + [x^{-1}] - \left[\frac{a-1}{a-x}\right] + \left[\frac{(1-a)x}{x-a}\right] = 2c_F, \end{aligned}$$

$$\bar{d}\left(\begin{bmatrix} a \\ x \end{bmatrix}\right) = 1, \qquad \rho\left(-\bar{d}\left(\begin{bmatrix} a \\ x \end{bmatrix}\right)\right) = -2c_F.$$

Combining the homomorphism $H_3(\mathrm{GL}_3(F)) \to H_3((\mathscr{D}_*)_{\mathrm{GL}_3(F)})$ and the homomorphism ρ, we get the mapping $\partial: H_3(\mathrm{GL}_3(F)) \to \mathfrak{p}(F)$.

LEMMA 3.4. *The restriction of the homomorphism just constructed to* $H_3(\mathrm{GL}_2(F))$ *coincides with the homomorphism* ∂ *in* §2.

PROOF. Denote by C'_p the subgroup of C^2_{p+1} generated by the generators $(x_0, \dots, x_{p+1})$, where

$$x_{p+1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Since d' does not affect the last point, C'_* is a $\mathrm{GL}_2(F)$-subcomplex of the complex $\mathscr{D}_*$. It is easy to see that the augmented complex $C'_* \xrightarrow{\varepsilon_{\mathscr{D}}} \mathbb{Z} \to 0$ obtained is acyclic. Consequently, the composition $H_3(\mathrm{GL}_2(F)) \to H_3(\mathrm{GL}_3(F)) \to H_3((\mathscr{D}_*)_{\mathrm{GL}_3(F)})$ can also be decomposed in the following form:

$$H_3(\mathrm{GL}_2(F)) \to H_3((C'_*)_{\mathrm{GL}_2(F)}) \to H_3((\mathscr{D}_*)_{\mathrm{GL}_3(F)}).$$

The projection $\mathbb{P}^2(F) \to \mathbb{P}^1(F)$ with center at the point $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ together with discarding of the last point determines a homomorphism of $\mathrm{GL}_2(F)$-complexes over $\mathbb{Z}: C'_* \to C_*$. The composition

$$H_3((\mathrm{GL}_2(F))) \to H_3((C'_*)_{\mathrm{GL}_2(F)}) \to H_3((C_*)_{\mathrm{GL}_2(F)}) = \mathfrak{p}(F)$$

clearly coincides with the homomorphism ∂ in §2. It now suffices to verify the commutativity of the diagram

$$\begin{array}{ccc} (C'_3)_{\mathrm{GL}_2(F)} & \longrightarrow & (C_3)_{\mathrm{GL}_2(F)} \\ \downarrow & & \downarrow \\ (\mathscr{D}_3)_{\mathrm{GL}_3(F)} & \xrightarrow{\rho} & \mathfrak{p}(F) \end{array}$$

Every orbit of the action of $\mathrm{GL}_2(F)$ on C'_3 contains a unique element of the form

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & a & 0 \\ * & * & * & * & 1 \end{pmatrix},$$

and the image of this element in $(C_4^2)_{\mathrm{GL}_3(F)} \subset (\mathscr{D}_3)_{\mathrm{GL}_3(F)}$ coincides with $\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]$ for some x, while its image in $\mathfrak{p}(F)$ is equal to $[a]$. The same answer is obviously obtained by going along a different path.

Proposition 3.1. *∂ determines an isomorphism*

$$H_3(\mathrm{GL}_3(F))/H_3(\mathrm{GM}_2(F)) + H_3(T_3) \xrightarrow{\sim} \mathrm{B}(F).$$

Proof. The group $H_3(T_3)$ lies in the kernel of ∂, since the augmentation $\varepsilon_{\mathscr{D}}: C_1^2 \to \mathbb{Z}$ has the T_3-equivalent section

$$n \mapsto n \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

On the other hand, $H_3(\mathrm{GL}_3(F))$ is generated by $H_3(\mathrm{GL}_2(F))$ and $H_3(T_3)$ (see [3]).

Corollary 3.1. *The kernel of the homomorphism $H_3(\mathrm{GL}_3(F)) \to H_3(\mathrm{GL}_3(F))$ is contained in $H_3(\mathrm{GM}_2(F))$.*

§4. Computation of $H_3(\mathrm{GL}(F))/H_3(\mathrm{GM}(F))$

According to [3], the imbedding $\mathrm{GL}_3(F) \hookrightarrow \mathrm{GL}(F)$ induces an isomorphism $H_3(\mathrm{GL}_3(F)) \xrightarrow{\sim} H_3(\mathrm{GL}(F))$, hence ∂ can be regarded as a homomorphism $H_3(\mathrm{GL}(F)) \to \mathrm{B}(F)$.

Lemma 4.1. $H_3(T) \subset \operatorname{Ker} \partial$.

Proof. It follows from the Künneth formula that $H_3(T)$ is generated by the subgroups $H_3(T^{ijk})$, where T^{ijk} is the subgroup of T consisting of the diagonal matrices nontrivial only in the positions i, j, and k. On the other hand, T^{ijk} is conjugate to T_3 by means of a permutation matrix, and hence the image of $H_3(T^{ijk})$ in $H_3(\mathrm{GL}(F))$ coincides with the image of $H_3(T_3)$. This shows that

$$\mathrm{Im}(H_3(T) \to H_3(\mathrm{GL}(F))) = \mathrm{Im}(H_3(T_3)) \subset \operatorname{Ker} \partial.$$

Let m and n be positive integers, and consider the group $\mathrm{GL}_m(F) \times \mathrm{GL}_n(F)$. Denote by p_1 and p_2 the projections on the factors, by q_i the homomorphism $\mathrm{GL}_m(F) \times \mathrm{GL}_n(F) \to \mathrm{GL}(F)$ equal to the composition of p_i and the imbedding $\mathrm{GL}_*(F) \hookrightarrow \mathrm{GL}(F)$, and by j the imbedding $\mathrm{GL}_m(F) \times \mathrm{GL}_n(F) \hookrightarrow \mathrm{GL}_{m+n}(F) \hookrightarrow \mathrm{GL}(F)$.

Lemma 4.2.

$$j_*([H^2(\mathrm{GL}_m(F)) \otimes H_1(\mathrm{GL}_n(F))] \oplus [H_1(\mathrm{GL}_m(F)) \otimes H_2(\mathrm{GL}_n(F))]) \subset H_3(T).$$

Proof. As is easy to see (for example, by using results in [3]), the imbedding $T_m \hookrightarrow \mathrm{GL}_m(F)$ induces an epimorphism onto H_1 and H_2 for any m. It remains to note that $j(T_m \times T_n) \subset T$.

COROLLARY 4.1. *$j_*(u)-(q_1)_*(u)-(q_2)_*(u) \in H_3(T)$ for any $u \in H_3(\mathrm{GL}_m(F) \times \mathrm{GL}_n(F))$.*

PROOF. For any groups G and H we denote by $\widetilde{H}_3(G\times H)$ the kernel of the homomorphism $((p_1)_*, (p_2)_*)\colon H_3(G \times H) \to H_3(G) \oplus H_3(H)$. The Künneth formula gives us the exact sequence

$$0 \to [H_1(G) \otimes H_2(H)] \oplus [H_2(G) \otimes H_1(H)] \to \widetilde{H}_3(G \times H) \to \mathrm{Tor}(H_1(G), H_1(H)) \to 0.$$

Since $H_1(\mathrm{GL}_*(F)) = H_1(\mathrm{GL}_1(F)) = F_*$, this exact sequence shows that

$$\widetilde{H}_3(\mathrm{GL}_m(F) \times \mathrm{GL}_n(F)) = H_1(\mathrm{GL}_m(F)) \otimes H_2(\mathrm{GL}_n(F)) + H_2(\mathrm{GL}_m(F)) \otimes H_1(\mathrm{GL}_n(F)) + \widetilde{H}_3(\mathrm{GL}_1(F) \times \mathrm{GL}_1(F)).$$

Thus, $j_*(\widetilde{H}_3(\mathrm{GL}_m(F) \times \mathrm{GL}_n(F))) \subset H_3(T)$. It now remains to note that $u - (p_1)_*(u) \otimes 1 - 1 \otimes (p_2)_*(u) \in \widetilde{H}_3(\mathrm{GL}_m(F) \times \mathrm{GL}_n(F))$ and

$$j_*((p_1)_*(u) \otimes 1) = (q_1)_*(u), \qquad j_*(1 \otimes (p_2)_*(u) = (q_2)_*(u).$$

LEMMA 4.3. $\mathrm{Im}(H_3^0(\mathrm{GM}(F)) \to H_3(\mathrm{GL}(F))) = \mathrm{Ker}\,\partial$.

PROOF. Let n be a sufficiently large positive integer. We consider the commutative diagram of group extensions

$$\begin{array}{ccccccccc} 1 & \to & T_2 & \to & \mathrm{GM}_2(F) \times \Sigma_{n-2} & \to & \Sigma_2 \times \Sigma_{n-2} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & T_n & \to & \mathrm{GM}_n(F) & \longrightarrow & \Sigma_n & \longrightarrow & 1 \end{array}$$

and compare the corresponding Hochschild-Serre spectral sequences. The spectral sequence corresponding to the upper extension is denoted by E', and that corresponding to the lower extension by $E: E^2_{pq} = H_p(\Sigma_n, H_q(T_n))$. Denote by F_i^* the ith term in the decomposition $T_n = F^* \oplus \cdots \oplus F^*$; then

$$H_2(T_n) = \Lambda^2(T_n) = \coprod_{i=1}^{n} \Lambda^2(F_i^*) \oplus \coprod_{i<j} (F_i^* \otimes F_j^*).$$

The action of Σ_n transitively permutes the terms of both of the first and of the second sum, and, moreover, $\mathrm{Stab}(\Lambda^2(F_1^*)) = \Sigma_1 \times \Sigma_{n-1} \cong \Sigma_{n-1}$ and $\mathrm{Stab}(F_1^* \otimes F_2^*) = \Sigma_2 \times \Sigma_{n-2}$. Thus,

$$E^2_{p2} = H_p(\Sigma_{n-1}, \Lambda^2 F^*) \oplus H_p(\Sigma_2 \times \Sigma_{n-2}, F^* \otimes F^*).$$

Similarly, $E'^2_{p2} = H_p(\Sigma_{n-2}, \Lambda^2 F^*) \oplus H_p(\Sigma_2 \times \Sigma_{n-2}, F^* \otimes F^*)$. By stabilization of the homology of Σ_n (with trivial coefficients; see [12]), we conclude that $E'^2_{p2} = E^2_{p2}$ for $p \leq 5$ if n is sufficiently large. In precisely the same way, $E'^2_{p1} = E^2_{p1}$ for $p \leq 5$ if n is sufficiently large. Under these conditions $E'^{\infty}_{1,2} = E^{\infty}_{1,2}$ and $E'^{\infty}_{2,1} = E^{\infty}_{2,1}$, and hence

$$H_3^0(\mathrm{GM}_n(F)) = H_3(T_n) + \mathrm{Ker}(H_3(\mathrm{GM}_2(F) \times \Sigma_{n-2}) \to H_3(\Sigma_2 \times \Sigma_{n-2})).$$

Passing to the limit with respect to n and using Corollary 4.1, we conclude that the image of $H_3^0(\mathrm{GM}_n(F))$ in $H_3(\mathrm{GL}(F))$ is contained in $H_3(T) + H_3(\mathrm{GM}_2(F)) = \mathrm{Ker}\,\partial$. The reverse inclusion follows from Remark 2.2.

LEMMA 4.4. *The image of $H_3(\Sigma)$ in $B(F)$ under the homomorphism ∂ coincides with the cyclic subgroup generated by $2c_F$.*

PROOF. The homology groups of the permutation groups Σ_n were computed by Nakaoka [12]. In particular, $H_3(\Sigma) = \mathbb{Z}/12 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We actually need only the following uncomplicated stabilization result: $H_3(\Sigma_6) = H_3(\Sigma)$. It is clear from this fact that $H_3(\Sigma)$ can have only 2-primary, 3-primary, and 5-primary components. A 5-Sylow subgroup of Σ_6 coincides with the cyclic subgroup generated by a cycle σ of length five, and thus the homomorphism $\mathbb{Z}/5 = H_3(\langle\sigma\rangle) \to H_3(\Sigma_6)$ is an epimorphism onto the 5-primary component. Finally, σ is conjugate to σ^2 in Σ_6, and hence the imbedding $\langle\sigma\rangle \hookrightarrow \Sigma_6$ and the homomorphism $\langle\sigma\rangle \xrightarrow{2} \langle\sigma\rangle \hookrightarrow \Sigma_6$ induce one and the same mapping on homology. But multiplication by 2 in $\langle\sigma\rangle$ induces multiplication by 4 in $H_3(\langle\sigma\rangle)$, and thus the 5-primary component of $H_3(\Sigma_6)$ is annihilated by the number 3, i.e., is equal to zero.

SUBLEMMA 4.4.1. *The image of the 2-primary component of $H_3(\Sigma)$ in $B(F)$ is equal to zero.*

Indeed, denote a p-Sylow subgroup of Σ_n by $\Sigma_n(p)$. Then the 2-primary component of $H_3(\Sigma)$ is the image of $H_3(\Sigma_6(2))$. Moreover, $\Sigma_6(2) = \Sigma_4(2)\times\Sigma_2$. For any $u \in H_3(\Sigma_4(2) \times \Sigma_2)$ we have that $(q_2)_*(u) \in H_3(\Sigma_2) \subset H_3(\mathrm{GM}_2(F)) \subset \operatorname{Ker}\partial$, hence in view of Corollary 4.1 it suffices to verify that $H_3(\Sigma_4(2)) \subset \operatorname{Ker}\partial$. The group $\Sigma_4(2)$ is generated by the permutations $(1, 2)$, $(3, 4)$, and $(1, 3)\cdot(2.4)$. We carry out conjugation with the help of the matrix γ indicated below; then our permutation matrices take the form

$$\gamma = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad (1, 2)^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(3, 4)^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad ((1, 3)\cdot(2, 4))^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The matrices obtained lie in the subgroup $\left(\begin{smallmatrix} B_2 & * \\ 0 & \mathrm{GM}_2(F) \end{smallmatrix}\right)$, the homology of which coincides with that of $T_2 \times \mathrm{GM}_2(F)$ and hence lies in $\operatorname{Ker}\partial$ according to Corollary 4.1.

SUBLEMMA 4.4.2. *The image of the 3-primary components of $H_3(\Sigma)$ in $B(F)$ coincide with the cyclic subgroup generated by $2c_F$.*

Indeed, the 3-primary component of $H_3(\Sigma)$ coincides with the image of $H_3(\Sigma_6(3))$. Further, $\Sigma_6(3) = \Sigma_3(3) \times \Sigma_3(3)$, and the same argument as above shows that the image of $H_3(\Sigma_6(3))$ in $B(F)$ coincides with the image of $H_3(\Sigma_3(3))$. The group $\Sigma_3(3)$ is generated by the ternary cycle $\tau = (1, 2, 3)$. If $\operatorname{char} F = 3$, then the permutation matrix τ is similar to an upper triangular matrix, and hence the homomorphism $H_3(\langle\tau\rangle) \to H_3(\mathrm{GL}(F))$ is zero [16]. Moreover, in this case the equation $x^2 - x + 1 = 0$ is solvable in F, therefore, $2c_F = 0$. We assume thus that $\operatorname{char} F \neq 3$. In this case τ is similar to the matrix $\lambda = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$. We construct a λ-equivalent homomorphism from the periodic resolution of $\mathbb{Z}$ to C_*:

$$\begin{array}{ccccccc}
\mathbb{Z}[\lambda] & \xleftarrow{1-\lambda} & \mathbb{Z}[\lambda] & \xleftarrow{1+\lambda+\lambda^2} & \mathbb{Z}[\lambda] & \xleftarrow{1-\lambda} & \mathbb{Z}[\lambda] \\
\downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\
C_0 & \longleftarrow & C_1 & \longleftarrow & C_2 & \longleftarrow & C_3 .
\end{array}$$

As f_0 we take multiplication by $\binom{1}{0}$, as f_1 multiplication by $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, as f_2 multiplication by $\left(\begin{smallmatrix}0&1&1\\1&0&x\end{smallmatrix}\right)+\left(\begin{smallmatrix}1&0&1\\1&1&x\end{smallmatrix}\right)+\left(\begin{smallmatrix}1&1&1\\0&1&x\end{smallmatrix}\right)$, where $x \in F^* - 1$, and as f_3 multiplication by

$$-\begin{pmatrix}0&1&1&1\\1&0&x&y\end{pmatrix}-\begin{pmatrix}1&0&1&1\\1&1&x&y\end{pmatrix}-\begin{pmatrix}1&1&1&1\\0&1&x&y\end{pmatrix}+\begin{pmatrix}1&0&-x&1\\1&1&1-x&y\end{pmatrix}$$
$$+\begin{pmatrix}1&1&-x&1\\0&1&1-x&y\end{pmatrix}+\begin{pmatrix}0&1&-x&1\\1&0&1-x&y\end{pmatrix},$$

where $y \neq 0, 1, x, (x-1)/x$. The image of the canonical generator of $H_3(\langle\lambda\rangle)$ in $\mathrm{B}(F)$ coincides with the image in $\mathrm{B}(F)$ of the element determining the homomorphism f_3, i.e., with

$$-\left[\frac{x}{y}\right]-\left[\frac{y-1}{x-1}\right]-\left[\frac{y(x-1)}{x(y-1)}\right]+[x(1-y)]+\left[\frac{-y}{(1-x)(1-y)}\right]+\left[\frac{1-x}{-xy}\right].$$

Assume that $x \neq -1$, and let $y = x^{-1}$; then the expression under consideration takes the form $-[x]^2 - 2[-x^{-1}] + 2[x-1] + [(-x)^{-2}]$. After using Lemmas 1.2 and 1.3 and Lemma 4.5 below, we see that this expression is equal to $-2c_F$.

LEMMA 4.5 ([6]). $[x^2] = 2([x]+[-x]+[-1])$ *for any* $x \neq \pm 1$ *in the group* $\mathfrak{p}(F)$.

PROOF. We use the relation

$$[x]-[x^2]+[x]-\left[\frac{1-x^{-1}}{1-x^{-2}}\right]+\left[\frac{1-x}{1-x^2}\right]=0.$$

From it,

$$\begin{aligned}
[x^2] &= 2[x]-\left[\frac{x}{1+x}\right]+\left[\frac{1}{1+x}\right]=2[x]+2\left[\frac{1}{1+x}\right]-c_F\\
&= 2[x]-2[1+x]-c_F\\
&= 2[x]+2[-x]-3c_F = 2([x]+[-x])+\langle -1\rangle.
\end{aligned}$$

Accordingly, we have proved the following theorem.

THEOREM 4.1. $H_3(\mathrm{GL}(F))/H_3(\mathrm{GM}(F)) = \mathrm{B}(F)2c_F$.

§5. Basic results

Denote by $h\colon S^3 \to S^2$ the Hopf bundle. For any topological space X the composition with h determines a mapping $h^*\colon \pi_2(X) \to \pi_3(X)$ (the distinguished points are fixed and are omitted in the notation). In general the mapping h^* is not a homomorphism, but it will be a homomorphism if, for example, X is an H-space (since in this case the group operation in $\pi_*(X)$ is induced by the operation in X).

LEMMA 5.1. *Suppose that X is a simply connected space, and w_α is a system of generators of the group $\pi_2(X)$. Then the Hurewicz homomorphism $\pi_3(X) \to H_3(X)$ is surjective, and its kernel is generated by the elements $h^*(w_\alpha)$ and $w_\alpha * w_\beta$ ($*$ is the Whitehead product; see* [1], *Chapter* V, §4).

PROOF. Denote by $f_\alpha: S^2_\alpha \to X$ some representative of w_α and by Y the bouquet of the two-dimensional spheres S^2_α. The mappings f_α determine $f: Y \to X$; replacing X by the mapping cylinder of f, we can assume that f is an imbedding. By construction, $\pi_2(Y) \to \pi_2(X)$ is an epimorphism, and hence $\pi_2(X, Y) = 0$. According to the Hurewicz isomorphism theorem, the homomorphism $\pi_3(X, Y) \to H_3(X, Y)$ is bijective, and the assertion follows from the commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \pi_3(Y) & \to & \pi_3(X) & \to & \pi_3(X, Y) & \to & \pi_2(Y) & \to & \pi_2(X) & \to & 0 \\ \downarrow & & \downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 0 = H_3(Y) & \to & H_3(X) & \to & H_3(X, Y) & \to & H_2(Y) & \to & H_2(X) & \to & 0 \end{array}$$

and the well-known description of π_3 of a bouquet of two-dimensional spheres (see [1], Chapter V, §4).

COROLLARY 5.1. *Assume in addition that X is an H-space; then the sequence $\pi_2(X) \xrightarrow{h^*} \pi_3(X) \to H_3(X) \to 0$ is exact.*

Let Λ be an associative ring with identity. We denote by $E(\Lambda) \subset \mathrm{GL}(\Lambda)$ the group of elementary matrices and apply Corollary 5.1 to the space $BE(\Lambda)^+$. This space is simply connected, and $\pi_i(BE(\Lambda)^+) = K_i(\Lambda)$ for $i \geq 2$. Therefore, the exact sequence in Corollary 5.1 takes the form $K_2(\Lambda) \xrightarrow{h^*} K_3(\Lambda) \to H_3(E(\Lambda)) \to 0$.

LEMMA 5.2. *The homomorphism $h^*: K_2(\Lambda) \to K_3(\Lambda)$ coincides with multiplication by $l(-1) \in K_1(\mathbb{Z})$.*

PROOF. Denote by $S = \{S^0, S_1, \ldots\}$ a sphere spectrum (see [2] about spectra and related concepts). In view of the presence for any spectrum X of a canonical equivalence $S \wedge X \cong X$ the spectrum S is a ring spectrum, and an arbitrary spectrum X is an S-module. This determines canonical pairings of homotopy groups $\pi_i(S) \otimes \pi_j(X) \to \pi_{i+j}(X)$. Further, h determines an element $h \in \pi_1(X) = \mathbb{Z}/2$ that is a generator of this group. If $X = \{X_0, X_1, \ldots\}$ is a Ω-spectrum, then $\pi_*(X_0) = \pi_*(X)$, and the diagram

$$\begin{array}{ccc} \pi_2(X_0) & \xrightarrow{h^*} & \pi_3(X_0) \\ \downarrow\wr & & \downarrow\wr \\ \pi_2(X) & \xrightarrow{\bar{h}} & \pi_3(X) \end{array}$$

clearly commutes. Assume now that T is a ring spectrum and X a T-module.

The commutativity of the diagram

$$\begin{array}{ccc} S\wedge X & \xrightarrow{i\wedge 1_X} & T\wedge X \\ & \searrow & \downarrow{\mu} \\ & & X, \end{array}$$

where $i: S\to T$ is the identity and μ is multiplication, shows that $\overline{h}: \pi_i(X)\to\pi_{i+1}(X)$ coincides with multiplication by $i_*(\overline{h})\in\pi_1(T)$. In particular, we take T and X to be the spectra of the algebraic K-theory of the rings $\mathbb{Z}$ and Λ. We see that the homomorphism $h^*: K_2(\Lambda)\to K_3(\Lambda)$ coincides with multiplication by the element $i_*(\overline{h})\in K_1(\mathbb{Z})=\mathbb{Z}/2$. The morphism $i: S\to\mathbb{K}(\mathbb{Z})$ splits into the composition $S\xrightarrow{i_0} T\xrightarrow{j}\mathbb{K}(\mathbb{Z})$, where T is the spectrum corresponding to the permutation group ($T_0=\mathbb{Z}\times B\Sigma^+$), and j corresponds to the imbedding $\Sigma\hookrightarrow \mathrm{GL}(\mathbb{Z})$. Here i_0 is an equivalence, according to the Barratt-Priddy-Quillen theorem [14], and j induces an isomorphism on π_1. Consequently, $i_*(\overline{h})$ is a nontrivial element, i.e., $i_*(\overline{h})=l(-1)$.

COROLLARY 5.2. *For an arbitrary field F there is an exact sequence $K_2(F)\xrightarrow{l(-1)} K_3(F)\to H_3(\mathrm{SL}(F))\to 0$. If the group $K_2(F)$ is 2-divisible, then $K_3(F)\xrightarrow{\sim} H_3(\mathrm{SL}(F))$.*

Let G be a quasiperfect group with a direct sum (see [10]), and assume in addition that the projection $G\to G^{\mathrm{ab}}=G/[G,G]$ has a section s. We consider the homomorphism $[G,G]\times G^{\mathrm{ab}}\xrightarrow{i\times s} G\times G\xrightarrow{\oplus} G$ and the mapping $B([G,G])^+\times BG^{\mathrm{ab}}\to BG^+$ induced by it.

LEMMA 5.3. *$B([G,G])^+\times BG^{\mathrm{ab}}\to BG^+$ is a homotopy equivalence.*

PROOF. The mapping induced on $\pi_1: G^{\mathrm{ab}}\to G^{\mathrm{ab}}$ is given by the formula $t\mapsto(1\oplus s(t))\,\mathrm{mod}[G,G]$, and since $1\oplus s(t)$ is conjugate to t, this mapping is the identity. The induced mapping on the universal coverings $B([G,G])^+\to B([G,G])^+$ coincides with the mapping induced by the homomorphism $\varphi: [G,G]\to[G,G]: g\mapsto g\oplus 1$. Since for any finite family $\{g_i\}$ of elements of $[G,G]$ there exists an $x\in[G,G]$ such that $g_i\oplus 1=xg_ix^{-1}$, φ induces the identity mapping on integer homology, and hence $B\varphi^+$ is a homotopy equivalence.

We consider the group $\mathrm{GM}(F)$. Its commutator subgroup M coincides with the group of monomial matrices having determinant 1 and an even permutation. It is easy to see that the group M is perfect, so that we can consider the space $\mathrm{BGM}(F)^+$. The imbedding $\mathrm{GM}(F)\hookrightarrow\mathrm{GL}(F)$ induces a morphism $\mathrm{BGL}(F)^+\to\mathrm{BGL}(F)^+$.

LEMMA 5.4. *The Hurewicz homomorphism determines an isomorphism*

$$K_3(F)/\pi_3(\mathrm{BGM}(F)^+)\xrightarrow{\sim} H_3(\mathrm{GL}(F))/H_3(\mathrm{GM}(F)).$$

PROOF. Applying Corollary 5.1 to the spaces $\mathrm{BSL}(F)^+$ and BM^+, we get the commutative diagram

$$\begin{array}{ccccccc} \pi_2(\mathrm{BM}^+) & \longrightarrow & \pi_3(\mathrm{BM}^+) & \longrightarrow & H_3(M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K_2(F) & \longrightarrow & K_3(F) & \longrightarrow & H_3(\mathrm{SL}(F)) & \longrightarrow & 0 \end{array}$$

with exact rows. Since $\pi_i(\mathrm{BM}^+) = \pi_i(\mathrm{BGM}(F)^+)$ $(i \geq 2)$ and the homomorphism $\pi_i(\mathrm{BGM}(F)^+) \to K_i(F)$ is surjective for $i \leq 2$, we conclude that $K_3(F)/\pi_3(\mathrm{BGM}(F)^+) = H_3(\mathrm{SL}(F))/H_3(M)$. Denote by $\mathrm{SM}(F)$ the group of monomial matrices with determinant 1. The projection $\mathrm{SM}(F) \to \mathrm{SM}(F)^{\mathrm{ab}} = \mathbb{Z}/2$ has a section determined by the matrix

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

According to Lemma 5.3 and the Künneth formula, we get that $H_3(\mathrm{SM}(F)) = H_3(M) \oplus (H_2(M) \otimes \mathbb{Z}/2) \oplus H_3(\mathbb{Z}/2)$. Consider the image of $H_3(\mathrm{SM}(F))$ in $H_3(\mathrm{SL}(F))$. The image of the second term falls in the range of the homomorphism $H_3(\mathrm{SL}(F)) \otimes H_1(\mathrm{SL}(F)) \to H_3(\mathrm{SL}(F))$ induced by the direct sum operation, and hence is equal to zero. The matrix α is similar to the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and thus the image of the third term is contained in the image of $H_3(M)$. Therefore, $H_3(\mathrm{SL}(F))/H_3(M) = H_3(\mathrm{SL}(F))/H_3(\mathrm{SM}(F))$.

Applying Lemma 5.3 to the group $\mathrm{GL}(F)$ and using the Künneth formula, we get the decomposition $H_3(\mathrm{GL}(F)) = H_3(\mathrm{SL}(F)) \oplus (H_2(\mathrm{SL}(F)) \otimes F^*) \oplus H_3(F^*)$. Since the last two terms lie in $H_3(\mathrm{GM}(F))$, the homomorphism $H_3(\mathrm{SL}(F))/H_3(\mathrm{SM}(F)) \to H_3(\mathrm{GL}(F))/H_3(\mathrm{GM}(F))$ is surjective. The homomorphism $\mathrm{GL}(F) \to \mathrm{SL}(F)$: $\alpha \to \alpha \oplus \det \alpha^{-1}$ carries $\mathrm{GM}(F)$ into $\mathrm{SM}(F)$, and hence enables us to define a mapping $H_3(\mathrm{GL}(F))/H_3(\mathrm{GM}(F)) \to H_3(\mathrm{SL}(F))/H_3(\mathrm{SM}(F))$ that is a left inverse to the mapping considered above. Thus, $H_3(\mathrm{SL}(F))/H_3(\mathrm{SM}(F)) \xrightarrow{\sim} H_3(\mathrm{GL}(F))/H_3(\mathrm{GM}(F))$.

The next result follows from Theorem 4.1 and Lemma 5.4.

THEOREM 5.1. $K_3(F)/\pi_3(\mathrm{BGM}(F)^+) = \mathrm{B}(F)/2c_F$.

COROLLARY 5.3. *The group* $\mathrm{B}(\mathbb{Q})$ *is a cyclic group of sixth order generated by the element* $c_{\mathbb{Q}}$.

PROOF. It is known (see [8]) that $K_3(\mathbb{Q}) = K_3(\mathbb{Z})$ is a cyclic group of 48 elements. Further, $\pi_3(B\Sigma^+) = \mathbb{Z}/24$ is mapped monomorphically into $K_3(\mathbb{Q})$. Consequently, $\mathrm{B}(\mathbb{Q})/2c_{\mathbb{Q}} = K_3(\mathbb{Q})/\pi_3(\mathrm{BGM}(\mathbb{Q})^+)$ consists of at most two elements. On the other hand, according to Proposition 1.1, the image of $c_{\mathbb{Q}}$ in $\mathrm{B}(\mathbb{Q})/2c_{\mathbb{Q}}$ is not equal to zero. Thus, $\mathrm{B}(\mathbb{Q})$ is generated by $c_{\mathbb{Q}}$, and it remains to use Proposition 1.1 once more.

Denote by $\pi_i^0(\mathrm{BGM}(F)^+)$ the kernel of the natural projection $\pi_i(\mathrm{BGM}(F)^+) \to \pi_i(\mathrm{B}\Sigma^+) = k_i$. For each i there is a canonical decomposition $\pi_i(\mathrm{BGM}(F)^+) = \pi_i^0(\mathrm{BGM}(F)^+) \oplus \pi_i(\mathrm{B}\Sigma^+)$. Repeating almost word-for-word the arguments given above, we see that $K_3(F)/\pi_3^0(\mathrm{BGM}(F)^+) = H_3(\mathrm{GL}(F))/H_3^0(\mathrm{GM}(F))$, and thus the following proposition is a consequence of Lemma 4.3.

PROPOSITION 5.1. $K_3(F)/\pi_3^0(\mathrm{BGM}(F)^+) = \mathrm{B}(F)$.

We now occupy ourselves with a description of the image of $\pi_3^0(\mathrm{BGM}(F)^+)$ in $K_3(F)$. Since direct sums and tensor products of monomial matrices are again monomial, the Loday multiplication [10] determines a ring structure on $\pi_*(\mathrm{BGM}(F)^+)$, and the homomorphisms $\pi_*(\mathrm{BGM}(F)^+) \to K_*(F)$ and $\pi_*(\mathrm{BGM}(F)^+) \to \pi_*(B\Sigma^+)$ are compatible with the multiplication. The group $\pi_i(\mathrm{BGM}(F)^+) = H_1(\mathrm{GM}(F))$ coincides with $F^* \oplus \mathbb{Z}/2$, and $F^* = \pi_1^0(\mathrm{BGM}(F)^+)$, $\mathbb{Z}/2 = \pi_1(B\Sigma^+)$. The homomorphism $\pi_1(\mathrm{BGM}(F)^+) \to K_1(F) = F^*$ is the identity on F^* and carries the generator $\overline{h}$ of the group $\pi_1(B\Sigma^+) = \mathbb{Z}/2$ into $-1 \in F^*$. From this it is clear that the image of $\pi_3^0(\mathrm{BGM}(F)^+)$ contains the decomposable part $K_3(F)_{\mathrm{dec}} = \mathrm{Im}(K_3^M(F) \to K_3(F))$ of the group $K_3(F)$. Since the image of the decomposable part of $\pi_*(B\Sigma^+)$ is obviously contained in the decomposable part of $K_*(F)$, it follows that the image of $\pi_3^0(\mathrm{BGM}(F)^+)$ in $K_3(F)_{\mathrm{nd}} = K_3(F)/K_3(F)_{\mathrm{dec}}$ coincides with the image of $\pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}} = \mathrm{Ker}(\pi_3(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to \pi_3(B\Sigma^+)_{\mathrm{nd}})$.

Denote by $X = \{X_0, X_1, \dots\}$ the Ω-spectrum obtained by applying the Segal machinery [14] to the Γ-space $\coprod_{n\geq 0} \mathrm{BGM}_n(F)$. The space X_0 coincides with $\mathbb{Z} \times \mathrm{BGM}(F)^+$, and, moreover, the Barratt-Priddy-Quillen theorem (see [14]), shows that X is equivalent to the suspension-spectrum $\mathscr{E}(BF_+^*)$ (the "plus" sign means that a distinguished point was added to the space). The indicated equivalence is determined by the function $\mathscr{E}(BF_+^*) \to X$, which coincides on the zero terms of the spectra with the imbedding $BF_* \hookrightarrow 1 \times \mathrm{BGM}(F^+)$. We identify $\mathrm{BGM}(F)^+$ with the subspace $0 \times \mathrm{BGM}(F)^+ \subset \mathrm{BGM}(F)^+$. The group structure on BF^* induces a ring structure on the spectra $\mathscr{E}(BF_+^*)$ and X, and the equivalence $\mathscr{E}(BF_+^*) \xrightarrow{\sim} X$ is compatible with multiplication by the corresponding spectra. Moreover, the induced multiplication on $\mathrm{BGM}(F)^+$ coincides with the multiplication of Loday.

Let us consider the Atiyah-Hirzebruch spectral sequence

$$E_{pq}^2 = H_p(F^*, k_q) \Rightarrow \pi_{p+q}^s(BF_+^*) = \pi_{p+q}(\mathbb{Z} \times \mathrm{BGM}(F)^+),$$

where k_* is the coefficient ring in the stable homology theory: $k_q = \pi_q^s(S^0) = \pi_q(\mathbb{Z} \times B\Sigma^+)$, $k_0 = \mathbb{Z}$, $k_1 = \mathbb{Z}/2$ with generator $\overline{h}$, $k_2 = \mathbb{Z}/2$ with generator $\overline{h}^2$, and $k_3 = \mathbb{Z}/24$. The group $\pi_2^s(BF_+^*) = \pi_2(\mathrm{BGM}(F)^+) = H_2(M)$ is easy to compute; it is equal to $k_2 \oplus (F^* \otimes F^*)_\sigma$. Since $E_{1,1}^2 = H_1(F^*, \mathbb{Z}/2) = F^* F^{*2}$ and the homomorphism $E_{1,1}^2 \to \pi_2^s(BF_+^*)/k_2 = (F^* \otimes F^*)_\sigma$ (given by the formula $x \mapsto x \otimes x$) is injective, the differential $E_{3,0}^2 \to E_{1,1}^2$ is zero. Finally, note that the spectral sequence under consideration has a k_*-module structure, the term $E_{1,2}^2 = H_1(F^*, k_2) = F^k \otimes k_2$ consists of decomposable elements, and the factor of $E^2_{2,1} = H_2(F^*, k_1)$ by the decomposable elements coincides with $\mathrm{Tor}(F^*, k_1) = \mu_2(F)$. From these remarks it is clear that we have the exact sequence

$$\mu_2(F) \to \pi_3(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to H_3(F^*)_{\mathrm{nd}} \oplus (k_3)_{\mathrm{nd}} \to 0,$$

and hence also the exact sequence

$$\mu_2(F) \to \pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to H_3(F^*)_{\mathrm{nd}} \to 0.$$

To compute $H_3(F^*)_{\mathrm{nd}}$ we use the following lemma.

LEMMA 5.5. *For an arbitrary Abelian group A there is the exact sequence*

$$0 \to \bigwedge^3(A) \to H_3(A) \to \mathrm{Tor}(A, A)^{-\sigma} \to 0,$$

where σ is the involution induced by transposition of arguments, the first homomorphism is induced by the homology multiplication, and the second has the form

$$H_3(A) \xrightarrow{\Delta_*} H_3(A \oplus A) \to \mathrm{Tor}(A, A).$$

PROOF. If A is a cyclic group, then the proof is obtained by direct verification. If the assertion is true for A_1 and A_2, then it is clear from the Künneth formula that it is true also for $A_1 \oplus A_2$. The theorem is thus valid for finitely generated groups. The general case is obtained by passing to the limit.

COROLLARY 5.4. $H_3(F^*)_{\mathrm{nd}} = \mathrm{Tor}(F^*, F^*) = \mathrm{Tor}(\mu(F), \mu(F))$.

Consider the homomorphism $\mu(F) \to M$ given by the formula $\xi \mapsto \mathrm{diag}(\xi, \xi^{-1})$. It induces the mapping

$$\begin{aligned}\mathrm{Tor}(\mu(F), \mu(F)) &= H_3(\mu(F)) \to H_3^0(M)\\ &= \pi_3^0(\mathrm{BGM}(F)^+)/\overline{h}\cdot\pi_2^0(\mathrm{BGM}(F)^+) \to \pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}}.\end{aligned}$$

LEMMA 5.6. *The composition* $\mathrm{Tor}(\mu(F), \mu(F)) \to \pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to H_3(F^*)_{\mathrm{nd}} = \mathrm{Tor}(\mu(F), \mu(F))$ *coincides with multiplication by two.*

PROOF. The homomorphism $\pi_*(\mathbb{Z} \times \mathrm{BGM}(F)^+) \to H_*(F^*)$ coincides with the Hurewicz homomorphism $\pi_*(X) \to H_*(X) = H_*(\mathscr{E}(BF^*_+)) = \widetilde{H}_*(BF^*_+) = H_*(F^*)$, and it can be decomposed into the composition $\pi_*(\mathbb{Z} \times \mathrm{BGM}(F)^+) \to \widetilde{H}_*(\mathrm{BGM}(F)^+ \times \mathbb{Z}) \to H_*(X) = H_*(F)^*$. We compute the restriction of the resulting homomorphism $\widetilde{H}_*(\mathrm{GM}(F)) = \widetilde{H}_*(\mathrm{BGM}(F)^+) \to H_*(F^*)$ to the homology of the torus T_n.

SUBLEMMA 5.1.1. *For $i > 0$ the homomorphism $H_i(T_n) \to H_i(F^*)$ coincides with $(p_1)_* + \cdots + (p_n)_*$, where $p_j\colon T_n \to F^*$ is the projection on the jth factor.*

PROOF. The mapping $BT_n \to \mathrm{BGM}(F)^+$ determines a morphism $\mathscr{E}((BT_n)_+) \to X$ of spectra that coincides (in the homotopy category of spectra) with the sum of morphisms $\mathscr{E}((BT_n)_+) \xrightarrow{\mathscr{E}(Bp_j)} \mathscr{E}(BF^*_+) \to X$, and hence the homomorphism induced by it on homology is the sum of the homomorphisms induced by the terms. It thus suffices to verify the assertion for $n = 1$. The morphism

$\mathscr{E}(BF_+^*) \to X$ is the sum of the equivalence described above and the trivial morphism corresponding to the mapping $BF^* \to \mathrm{pt} \to -1 \times \mathrm{BGM}(F)^+$. It remains to see that the latter morphism induces the zero mappings on H_i for $i > 0$.

According to Sublemma 5.1.1, the composition $\mathrm{Tor}(\mu(F), \mu(F)) \to \mathrm{Tor}(\mu(F), \mu(F))$ coincides with the homomorphism

$$H_3(\mu(F)) \to H_3(\mu(F) \times \mu(F)) \xrightarrow{(p_1)_* + (p_2)_*} H_3(\mu(F)),$$

where the first mapping is induced by the homomorphism $\xi \to \xi \times \xi^{-1}$, and hence is equal to $1 + (-1)_* = 2$.

LEMMA 5.7. *The homomorphism* $\mathrm{Tor}(\mu(F), \mu(F)) \to \pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to K_3(F)_{\mathrm{nd}}$ *is injective.*

PROOF. Denote by $\overline{F}$ the algebraic closure of the field F, and by μ the group of roots of unity in $\overline{F}$. Since the homomorphism $\mathrm{Tor}(\mu(F), \mu(F)) \to \mathrm{Tor}(\mu, \mu)$ is injective, we can assume that $F = \overline{F}$. To prove the lemma we use the theory of Chern classes [15]. Let n be an integer not divisible by $\mathrm{char}\, F$. The group $K_4(F, \mathbb{Z}/n)$ is equal to ${}_nK_3(F)$ (since $K_4(F)$ is divisible ([17], [18])), and hence is equal to ${}_n(K_3(F)_{\mathrm{nd}})$ (since $K_3^M(F)$ is uniquely divisible [4]). The Chern class $c_{2,0}\colon K_4(F, \mathbb{Z}/n) \to H^0(F, \mu_n^{\otimes 2})$ thus takes the form $c_{2,0}\colon {}_n(K_3(F)_{\mathrm{nd}}) \to \mu_n^{\otimes 2}$. The composition $c_{2,0}$ with the homomorphism $\mathrm{Tor}(\mu_n, \mu_n) =_n \mathrm{Tor}(\mu, \mu) \to_n (K_3(F)_{\mathrm{nd}})$ coincides (see the construction of c_{ij} in [15]) with the homomorphism $H_4(\mu_n, \mathbb{Z}/n) =_n [H_3(\mu_n)] = \mathrm{Tor}(\mu_n, \mu_n) \to \mu_n^{\otimes 2}$ corresponding to multiplication by the Chern class [8] $c_2(\rho) \in H^4(\mu_n, \mu_n^{\otimes 2})$ of the representation $\rho\colon \mu_n \to \mathrm{GL}_2(F)$; $\rho(\xi) = \mathrm{diag}(\xi, \xi^{-1})$.

The representation ρ is a sum of two one-dimensional representations, and the multiplicativity of the full Chern class shows that $c_2(\rho) = -c_1^2$, where $c_1 \in H^2(\mu_n, \mu_n)$ is the Chern class of the representation $\mu_n \hookrightarrow F^*$. Since $c_1(\mathrm{id}_{F^*}) \in H^2(F^*, \mu_n)$ is a generator of this group corresponding to the identity mapping $H_2(F^*, \mathbb{Z}/n) =_n F^* \xrightarrow{\sim} \mu_i$, it follows that c_1 is a generator of $H^2(\mu_n, \mu_n)$. Since the cohomology of the cyclic group given by multiplication by a generator of the two-dimensional cohomology group is 2-periodic, we conclude that $-c_1^2$ is a generator of $H^4(\mu_n, \mu_n^{\otimes 2})$, and hence multiplication by $c_2(p) = -c_1^2\colon H_4(\mu_n, \mathbb{Z}/n) \to \mu_n^{\otimes 2}$ is an isomorphism.

COROLLARY 5.5. *There is the exact sequence*

$$0 \to \mu_2(F) \to \pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to \mathrm{Tor}(\mu(F), \mu(F)) \to 0$$

and if $\mathrm{char}\, F \neq 2$, *then the extension obtained is nontrivial.*

PROOF. It can be assumed that $\mathrm{char}\, F \neq 2$. The homomorphism $\pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to \mathrm{Tor}(\mu(F), \mu(F))$ has in view of Lemmas 5.6 and 5.7 a nontrivial kernel, and this proves the first assertion. To prove the second one it suffices to see that the 2-torsion in $\pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}}$ consists of two elements. It is now clear from what has been proved that $\pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}}$ is mapped injectively upon extension of the field, and for an algebraically closed F the homomorphism $\operatorname{Tor}(\mu(F), \mu(F)) \to \pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}}$ is bijective in view of Lemmas 5.6 and 5.7.

If G is a cyclic group with nontrivial 2-torsion, then $\widetilde{G}$ denotes the unique nontrivial extension of G by $\mathbb{Z}/2$. Passing to the limit (or computing Ext directly), we define $\widetilde{G}$ also for an ind-cyclic group G. There is always a canonical injective homomorphism $G \hookrightarrow \widetilde{G}$ whose composition with the projection $\widetilde{G} \to G$ coincides with multiplication by two. This homomorphism is bijective if and only if the group G is 2-divisible. If the 2-torsion in G is equal to zero, then we set $\widetilde{G} = G$. According to Lemma 5.5, the group $\pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}}$ coincides with $\operatorname{Tor}(\mu(F), \mu(F))^{\sim}$.

LEMMA 5.8. *The homomorphism* $\pi_3^0(\mathrm{BGM}(F)^+)_{\mathrm{nd}} \to K_3(F)_{\mathrm{nd}}$ *is injective.*

PROOF. Since $\operatorname{Tor}(\mu(F), \mu(F))^{\sim}$ is mapped injectively upon extension of F, F can be assumed to be algebraically closed. In this case the assertion follows from Lemma 5.7.

THEOREM 5.2. *For any infinite field F there is the exact sequence*

$$0 \to \operatorname{Tor}(\mu(F), \mu(F)^{\sim}) \to K_3(F)_{\mathrm{nd}} \to \mathrm{B}(F) \to 0.$$

COROLLARY 5.6. *The Bloch group is rationally invariant.*

COROLLARY 5.7. *If F is algebraically closed, then* $\mathrm{B}(F)$ *is uniquely divisible.*

PROOF. It is known ([17], [18]) that the group $K_3(F)$ is divisible, and the torsion subgroup is isomorphic to $\coprod_{l \neq \operatorname{char} F} \mathbb{Q}_l/\mathbb{Z}_l$; the same applies to the group $K_3(F)_{\mathrm{nd}}$, since $K_3^M(F)$ is uniquely divisible [4]. On the other hand, $\operatorname{Tor}(\mu(F), \mu(F))^{\sim} \cong \coprod_{l \neq \operatorname{char} F} \mathbb{Q}_l/\mathbb{Z}_l$. Since every injective mapping of $\mathbb{Q}_l/\mathbb{Z}_l$ into itself is clearly bijective, we conclude that $\operatorname{Tor}(\mu(F), \mu(F))^{\sim}$ is mapped bijectively onto the torsion subgroup of $K_3(F)_{\mathrm{nd}}$.

REMARK 5.1. It is possible to give a more direct proof of Corollary 5.7 that does not use the results in [17] and [18]. In investigating this case in particular the author found a general approach to the proof of the Quillen-Lichtenbaum conjecture that was later developed in [17] and [7]. We give a sketch of this proof.

We begin with a construction of the specialization homomorphisms. Let v be a discrete valuation of the field F with valuation ring $\mathcal{O}$ and residue field k. Choose a local parameter π and define the homomorphisms $s_\pi\colon (F^* \otimes F^*)_\sigma \to (k^* \otimes k^*)_\sigma$, $s_\pi\colon \mathfrak{p}(F) \to \mathfrak{p}(k)$ by the formulas

$$s_\pi(x \otimes y) = \overline{\left(\frac{x}{\pi^{v(x)}}\right)} \otimes \overline{\left(\frac{y}{\pi^{v(y)}}\right)}, \qquad s_\pi([x]) = \begin{cases} [\overline{x}] & \text{if } x \in \mathcal{O}, \\ \langle \overline{\frac{x}{\pi^{v(x)}}} \rangle & \text{if } x \notin \mathcal{O}, \end{cases}$$

where, by convention, $[0] = [1] = 0$. It is easy to see that these definitions are unambiguous, and the diagram

$$\begin{array}{ccc} \mathfrak{p}(F) & \longrightarrow & (F^* \otimes F^*)_\sigma \\ \downarrow s_\pi & & \downarrow s_\pi \\ \mathfrak{p}(k) & \longrightarrow & (k^* \otimes k^*)_\sigma \end{array}$$

is commutative. A homomorphism $\mathrm{B}(F) \to \mathrm{B}(k)$ thus arises that is easily seen not to depend on the choice of π.

It is easy to verify that $\mathrm{B}(F)$ is uniquely divisible (for an algebraically closed F) if and only if the group $\mathfrak{p}(F)$ is uniquely divisible. We confine ourselves to unique divisibility by positive integers n not divisible by $\operatorname{char} F$. Then n-divisibility of $\mathfrak{p}(F)$ is known from work of Dupont and Sah [6]; it follows from the formula $[x^n] - n(\sum_{\xi\in\mu_n}[\xi x] - \sum_{\xi\in\mu_n}[\xi]) = 0$. To prove this formula we consider the element $[t^n] - n(\sum_{\xi\in\mu_n}[\xi t] - \sum_{\xi\in\mu_n}[\xi]) \in \mathrm{B}(F(t))$. Since $\mathrm{B}(F(t)) = \mathrm{B}(F)$, it follows that the specializations of this element at all rational points of $\mathbb{A}^1_F$ are equal to each other. However, specializing at the point $t = x$, we get the element under consideration, and specializing at the point $t = 1$, we get zero.

To prove that $\mathfrak{p}(F)$ is uniquely n-divisible we define the homomorphism $\mathfrak{p}(F) \to \mathfrak{p}(F)$ inverse to multiplication by n with the help of the formula $[x] \to \sum_{y^n=x}[y] - \sum_{\xi\in\mu_n}[\xi]$. To prove that this is unambiguous we must verify the relation

$$\sum_{\xi\in\mu_n}\left([\xi a] - [\xi b] + \left[\xi\frac{b}{a}\right] - \left[\xi\frac{bc}{a}\right] + [\xi c] - [\xi]\right) = 0,$$

where a, b, and c are connected by the relation $c^n(1-b^n) = 1-a^n$ $(a^n \neq b^n \in F^* - 1)$. Denote the left-hand side of this equality by $f(a, b, c)$. We now fix the element $c \in F^*$ $(c \notin \mu_n)$ and consider the smooth affine curve X given by the equation $c^n(1-B^n) = 1-A^n$ along with its projective model $\overline{X}$. The element $f(A, B, c)$ lies by construction in ${}_n\mathfrak{p}(F(X))$, and, moreover, it is not hard to verify that it lies in $\mathrm{B}(F(X))$. The specialization of this element at the point $A = a$, $B = b$ coincides with $f(a, b, c)$. If $D = \Sigma n_i x_i$ is an arbitrary divisor on $\overline{X}$, then we define the specialization homomorphism $s_D\colon \mathrm{B}(F(X)) \to \mathrm{B}(F)$ as $\Sigma n_i s_{x_i}$, where s_{x_i} is the specialization homomorphism at the point x_i. It follows easily from the invariance of the Bloch group that $s_D = 0$ if the divisor D is a principal divisor (the transfer homomorphisms (see below) are needed for the proof). Moreover, $s_{nD}(f(A, B, C)) = 0$ for any D, since $n \cdot f(A, B, c) = 0$. The divisibility of the Picard group $\operatorname{Pic}\overline{X}$ shows that any divisor of degree zero on $\overline{X}$ is a sum of a principle divisor and a divisor divisible by n, therefore, $s_x(f(A, B, c)) = s_y(f(A, B, c))$ for any points x and y. It remains to see that the specialization $f(A, B, c)$ is equal to zero at the point $A = B = 1$.

REMARK 5.2. Denote by $G\mu$ the subgroup of $\mathrm{GM}(F)$ consisting of the monomial matrices whose elements are roots of unity. It is easy to see that we have the exact sequence $\pi_3^0(\mathrm{BG}\mu^+) \to K_3(F)_{\mathrm{nd}} \to \mathrm{B}(F) \to 0$. This enables us to define a transfer homomorphism $N_{E/F}\colon \mathrm{B}(E) \to \mathrm{B}(F)$ for a finite extension E/F such that in some basis of E/F all the elements of $\mu(E)$ are represented by monomial matrices. This condition is satisfied if, for example, $\mu(E) = \mu(F)$. However, the example of the extension $\mathbb{C}/\mathbb{R}$ shows that it is not true that a transfer homomorphism exists in an arbitrary finite extension (the existence of a transfer homomorphism implies the equality $[E:F] \times \operatorname{Ker}(\mathrm{B}(F) \to \mathrm{B}(E)) = 0$, which is not satisfied in the case of $\mathbb{C}/\mathbb{R}$).

REMARK 5.3. Corollary 5.7 can be strengthened considerably by using results in [11], where the torsion and cotorsion are computed in $K_3(F)_{\mathrm{nd}}$ for any field F. In particular, it can be shown that $\mathrm{B}(F)$ is uniquely divisible for any field containing an algebraically closed subfield. Moreover, for an arbitrary field F, if F_0 denotes the subfield of constants in F, then $\mathrm{B}(F_0) \hookrightarrow \mathrm{B}(F)$, and the factor group $\mathrm{B}(F)/\mathrm{B}(F_0)$ is uniquely divisible.

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