Cluster Algebras & the Amplituhedron

Lecture 1: What is the amplituhedron & what is a cluster algebra?

The **Grassmannian** $Gr_{k,n}(\mathbb{R}) = \{V \mid V \subset \mathbb{R}^n \mid \dim V = k\}$

Represent an element of $Gr_{k,n}$ by full rank $k \times n$ matrix C.

$$C = \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

Given $I \in \binom{[n]}{k}$, the **Plucker coord** $p_I(C)$ is the minor of C in colums I. Ex...

The **totally nonnegative** (TNN) (sometimes called "positive") **Grassmannian** $Gr_{k,n}^{\geq 0}$ is the subset of $Gr_{k,n}$ where $p_I \geq 0$ $\forall I$.

Background: 1994 Lusztig, 1997 Rietsch, 2006 Postnikov

One can partition $Gr_{kn}^{\geq 0}$ into pieces based on which Plucker coord's are pos & which are 0.

Let $\mathcal{M} \subseteq \binom{[n]}{k}$. Let $S_{\mathcal{M}} := \{C \in Gr_{kn}^{\geq 0} \mid p_I(C) > 0 \text{ iff } I \in \mathcal{M}\}$

Postnikov/Rietsch: Each nonempty $S_{\mathcal{M}}$ is a cell.

Postnikov-Speyer-W: Cells glue together nicely, ie.

$$Gr_{kn}^{\geq 0} = \coprod S_{\mathcal{M}} \quad \text{is a CW complex}$$

W: The face poset of cells (ordered by containment of closures) is thin & shellable $\Longrightarrow$ it is the face poset of a regular CW decomp of a ball.

Rietsch-W: For each positroid cell S_π, $\overline{S_\pi}$ is contractible, w/ bdy homotopy equiv to sphere.

Galashin-Karp-Lam $\overline{S_\pi}$ is homeomorphic to closed ball.

Note: The cells of $Gr_{kn}^{\geq 0}$ have been classified & are in bijection eg w/ decorated permutations (Postnikov).
+ equiv classes of plabic graphs

Now for amplituhedron... defined by Arkani-Hamed – Trnka (2013)

Def: Fix n, k, m w/ $k+m \leq n$.
Let $Z \in Mat^{>0}_{n,k+m}$ be $n \times (k+m)$ matrix w/ max'l minors pos.

$$\begin{array}{cc} & k+m \\ n & \begin{pmatrix} \quad & \quad \end{pmatrix} \end{array}$$

Let $\tilde{Z}: Gr_{kn}^{\geq 0} \longrightarrow Gr_{k,k+m}$ be map

$$k\overset{n}{\begin{pmatrix} C \end{pmatrix}} \mapsto k\overset{n}{\begin{pmatrix} C \end{pmatrix}} {}_{n}\overset{k+m}{\begin{pmatrix} Z \end{pmatrix}} = k\overset{k+m}{\begin{pmatrix} CZ \end{pmatrix}}$$

(where identify matrices w/ row spans)

Rk: Fact that Z has max'l minors pos $\Rightarrow$ $\tilde{Z}$ well-defined.

Let $\mathcal{A}_{n,k,m}(Z) := \tilde{Z}(Gr_{kn}^{\geq 0}) \subseteq Gr_{k,k+m}$

Motivation: ($N=4$ SYM)

- the BCFW recurrence (2005) expresses scatt amplitudes as sums of rat'l functions of momenta.
  Indiv terms have "spurious poles," – singularities not present in amplitude

- Hodges (2009) observed that in some cases, the amplitude is the volume of a polytope, w/ spurious poles arising from internal boundaries of triangulatn of polytope.

  Q: Is each amplitude the volume of geom object?
- AH-T: yes.

  BCFW recurrence is interpreted as "triangulatn" of $A_{n,k,4}(Z)$

$A_{n,k,m}(Z)$ generalizes lots of nice objects!

(1) If Z is a square matrix, ie, $k+m=n$, then

$$\tilde{Z} : (Gr_{k,n})_{\geq 0} \longrightarrow Gr_{k,n}(\mathbb{R})$$
$$A \longmapsto AZ^t \quad \text{is injective}$$

so $A_{n,k,m}(Z) \cong (Gr_{k,n})_{\geq 0}$

(2) If $k=1$ and $m=2$, $A_{n,k,m}(Z) \subset Gr_{1,3} = \mathbb{P}^2$ is a polygon in $\mathbb{P}^2$:

$$1\overset{n}{(a_1 : \cdots : a_n)}_{a_i \geq 0} \longmapsto (a_1 : \cdots : a_n)\,{}_n\!\left(\overset{3}{Z^t}\right) \subset \mathbb{P}^2$$

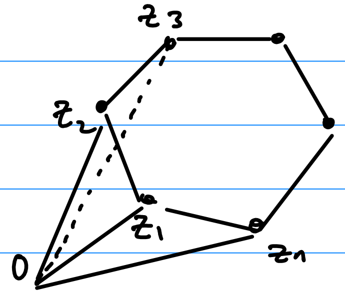
Let $e_i = (0 \dots \underset{\substack{\uparrow \\ i^{th} \text{ positn}}}{1} \dots 0)$, and write $Z = {}_3\overset{n}{\left(z_1 | z_2 | \cdots | z_n\right)}$.

Then $e_i \mapsto z_i$ (a point in $\mathbb{P}^2$)

The fact that max'l minors of Z are positive

$\Rightarrow$ the z_i's are in convex position

So as the a_i's vary over $\mathbb{R}_{\geq 0}$, $(a_1 : \cdots : a_n)\left(Z^t\right)$ gives all points in the cone spanned by the z_i's.



(3) If $k=1$, $\mathcal{A}_{n,k,m}(Z) \subset Gr_{1,m+1} = \mathbb{P}^m$ is combinatorially equivalent to cyclic polytope w/ n vertices in $\mathbb{P}^m$.

(Distinguished among simplicial polytopes for maximizing # faces, given dim & # vertices)

(4) If $m=1$, $\mathcal{A}_{n,k,m}(Z) \cong$ bounded complex of a cyclic hyperplane arrangement in $Gr_{k,k+1} \cong \mathbb{P}^k$ (Karp-W.)

(5) $m=2$ and 4 most interesting for physics...

Tiles & tilings of amplituhedron

Have $Gr_{kn}^{\geq 0} = \coprod_{\mathfrak{m}} S_{\mathfrak{m}}$ a cell cx, and

$$\underset{\text{dim } k(n-k)}{\tilde{Z} : Gr_{kn}^{\geq 0}} \twoheadrightarrow \underset{\text{dim } km}{\mathcal{A}_{n,k,m}(Z)} \quad \text{a surj map}$$

Def: If $\tilde{Z}$ is injective on km-dim'l cell $S_{\mathfrak{m}}$ we say $Z_{\mathfrak{m}} := \tilde{Z}(S_{\mathfrak{m}})$ is a tile for $\mathcal{A}_{n,k,m}(Z)$.
A tiling of $\mathcal{A}_{n,k,m}(Z)$ is a collection $\{Z_{\mathfrak{m}} \mid \mathfrak{m} \in \mathcal{C}\}$ of tiles such that:

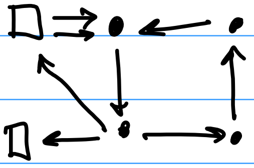
- their union equals $\mathcal{A}_{n,k,m}(Z)$
- their interiors are pairwise disjoint.

The "volume" of the amplituhedron computes scatt amplitudes
$\Longleftrightarrow$ a certain collection of BCFW cells in $Gr_{kn}^{\geq 0}$ give a tiling of $\mathcal{A}_{n,k,4}(Z)$. (Even-Zohar–Lakrec-Tessler)

Cluster Algebras: Defined by Fomin & Zelevinsky ~2000
Class of comm. rings w/ combinatorial structure.
Distinguished generators called cluster variables,
& relations encoded by quivers & quiver mutation.

Cluster varieties: Defined by Fock & Goncharov
"Varieties whose coord rings are cluster algebras"
Come w/ many torus charts (whose coord function are collection of clust var's)
Examples: Grassmannians, flag varieties, Schub varieties...

Def: A *quiver* is a finite directed graph.
Multiple edges allowed.
Oriented cycles of length 1 or 2 forbidden.
Two types of vertices: frozen & mutable.
Ignore edges between frozen vertices.
Let $n = \#$ of mutable vertices
$m =$ tot # of vertices



Quiver Mutation: Let k be a mutable vertex of Q.
We can *mutate* at k to produce new quiver $\mu_k(Q)$:
- For each instance of $j \to k \to l$, introduce edge $j \to l$
- Reverse direction of all edges incident to k
- Remove oriented 2-cycles

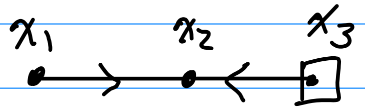
Rk: $\mu_k^2(Q) = Q \quad \forall k$ (involution)

Say 2 quivers are mutation-equiv if one can get between them via sequence of mutation.

<u>Def:</u> Let $\mathcal{F}$ be field of rat'l functions in m indep var's/$\mathbb{C}$. A <u>seed</u> in $\mathcal{F}$ is a pair $(Q, \underline{x})$ s.t.

- Q a quiver w/ n mutable vertices & $n-m$ frozen vert
- $\underline{x}$ is an extended cluster, an m-tuple of alg indep elemts of $\mathcal{F}$, indexed by vertices of Q.

<u>Ex:</u>



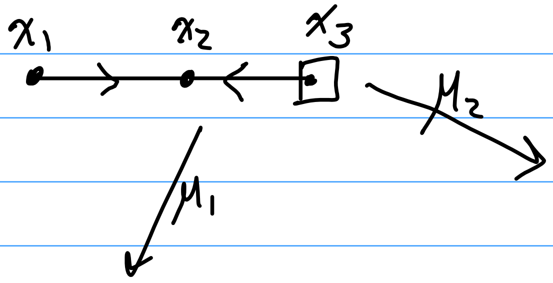
seed in $\mathcal{F} = \mathbb{C}(x_1, x_2, x_3)$

<u>Def</u> (Seed mutation). Let k be mutable vertex in Q & let x_k be the corresp cluster variable. The <u>seed mutation</u> $\mu_k : (Q, \underline{x}) \to (Q', \underline{x}')$ defined by:

- $Q' = \mu_k(Q)$
- $\underline{x}' = (\underline{x} \cup \{x_k'\}) \setminus \{x_k\}$ where

$$x_k x_k' = \prod_{\substack{j \\ j\to k}} x_k + \prod_{\substack{j \\ k\to j}} x_k \quad \text{(exchange rel)}$$

<u>Ex:</u>



Rk: Mutation is an involution.

Def: Let $\Sigma = (Q, \underline{x})$ be a seed, w/ extended cluster

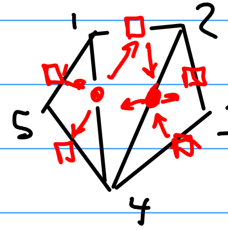
$$(\underbrace{x_1, \dots, x_n}_{\text{initial clust var}}, \underbrace{x_{n+1}, \dots, x_m}_{\text{frozen var's}})$$

Let $\mathcal{X}$ be (possibly infinite) set of all clust var's, obtained from $\underline{x}$ by all possible sequences of mutations.
Let $R = \mathbb{C}[x_{n+1}^{\pm 1}, \dots, x_m^{\pm 1}]$ be the <u>ground ring</u>

The cluster algebra $\mathcal{A}(\Sigma) := R[\mathcal{X}] \subset \mathcal{F}$ is the R-subalg generated by $\mathcal{X}$.

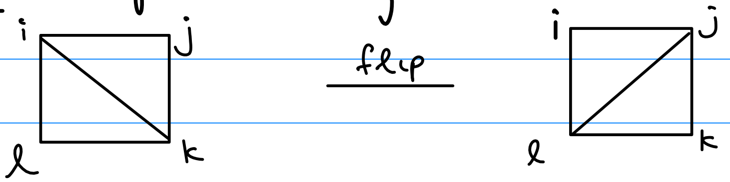
Ex: Grassmannian $Gr_{2,n}(\mathbb{C}) = \{ V \subset \mathbb{C}^n \mid \dim V = 2 \}$
Elements represented by full rk $2 \times n$ matrices M.
For $1 \le i < j \le n$, $p_{ij}(M) = \det \begin{pmatrix} m_{1i} & m_{1j} \\ m_{2i} & m_{2j} \end{pmatrix}$

Given triangulation T of n-gon we get <u>seed</u> $\Sigma(T)$



<u>Cluster</u> is $(p_{14}, p_{24}, \underbrace{p_{12}, p_{23}, p_{34}, p_{45}, p_{15}}_{\text{frozen}})$

Exercise: Flip of triangulation $\longleftrightarrow$ mutation



$(i < j < k < l)$

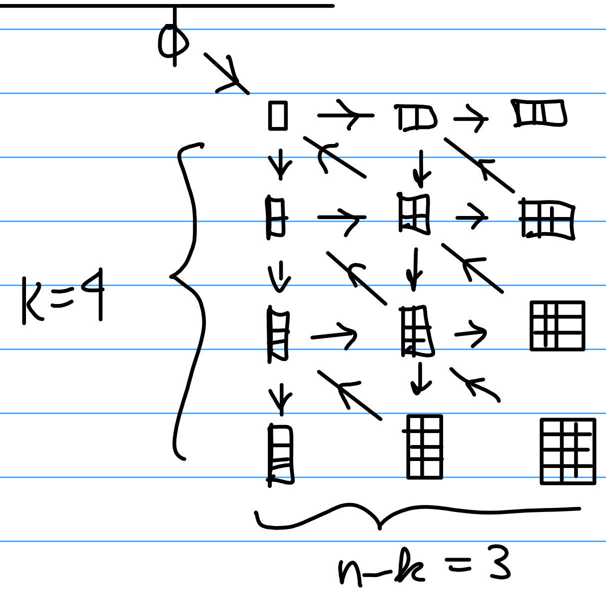
Exchange relation: $p_{ik} p_{jl} = p_{ij} p_{kl} + p_{il} p_{jk}$

Note: For any k,n, $\mathbb{C}[\widehat{Gr_{k,n}^\circ}]$ is a cluster alg (Scott).

Case of $Gr_{2,n}$: the cluster variables are exactly the set of Plucker coordinates

Case of $Gr_{k,n}$: Each Plucker coord is a clust var but there are many more clust var's (higher degree poly's in Pluckers)

Rectangles seed for $Gr_{k,n}$: is initial seed for clust alg



Identify each rectangle w/ k-element subset of $[n]$ & corresp Plucker coord

Q: Is amplituhedron connected to clust alg?

- Physicists observed that when one calculates scatt amplitudes as rat'l functions of momenta, the poles arising in expressions seemed to be related to compat collections of clust var's "cluster adjacency"

— Drummond – Foster – Gurdogan,
Lukowski – Parisi – Spradlin – Volovich

Lecture 2:

Recall: Fix n, k, m w/ $k+m \le n$.

Let $Z \in Mat^{>0}_{n,k+m}$ be $n \times (k+m)$ matrix w/ max'l minors pos.

$$\begin{array}{cc} & k+m \\ n & \left(\qquad\right) \end{array}$$

Let $\tilde{Z} : Gr^{\ge 0}_{kn} \longrightarrow Gr_{k,k+m}$ be map

$$k\left(\overset{n}{C}\right) \mapsto k\left(\overset{n}{C}\right) n\left(\overset{k+m}{Z}\right) = k\left(\overset{k+m}{CZ}\right)$$

(where identify matrices w/ row spans)

Let $A_{n,k,m}(Z) := \tilde{Z}\left(Gr^{\ge 0}_{kn}\right) \subseteq Gr_{k,k+m}$

Note: The cases we care most about are $m = 2, 4$ "small".
And $Gr_{k,k+m} \cong Gr_{m,k+m}$. This motivates following def of
B-amplituhedron

Def: (Karp-W): Choose $Z \in Mat^{>0}_{n,k+m}$ & let $W \in Gr^{>0}_{k+m,n}$ be the column span of Z. Define the B-amplituhedron

$$\mathcal{B}_{n,k,m}(W) := \{ V^{\perp} \cap W \mid V \in Gr^{\ge 0}_{kn} \} \subseteq Gr_m(W) \cong Gr_{m,k+m}$$

Prop (Karp-W): $\mathcal{B}_{n,k,m}(W)$ is homeomorphic to $A_{n,k,m}(Z)$. The map is

$$\begin{array}{rcl} f_Z : Gr_m(W) & \longrightarrow & Gr_{k,k+m} \\ X & \longmapsto & Z(X^{\perp}) \end{array}$$

$Z : \mathbb{R}^n \to \mathbb{R}^{k+m}$

So for small m this can be useful point of view

Coordinates for Amplituhedron

We'd like to be able to talk about $A_{n,k,m}(Z)$ directly inside $Gr_{k,k+m}$ ... need good coordinates

Def: (AH–Thomas) Fix Z as above, & let $Y \in Gr_{k,k+m}$ (thought of as matrix).
Write rows of Z as $\begin{pmatrix} Z_1 \\ \hline \vdots \\ \hline Z_n \end{pmatrix}$.

Given $I = \{i_1 < \dots < i_m\} \subset [n]$, define the *twistor coordinate*

$$\langle Y Z_I \rangle = \langle Y Z_{i_1} \dots Z_{i_m} \rangle := \det \begin{pmatrix} Y \\ \hline Z_{i_1} \\ \hline \vdots \\ \hline Z_{i_m} \end{pmatrix}$$

Lemma: (Kar-W) Twistor coords for A-amplituhedron equal Plucker coords for B-amp:
If $Y = f_Z(X)$ then $p_I(X) = \langle Y Z_I \rangle$ for all $I \in \binom{[n]}{m}$

Therefore we can identify twistor coordinates w/ Plucker coordinates in $Gr_{m,n}$ and poly's in twistor coord's ("functionaries") w/ poly's in Plucker coords, i.e. elements of $\mathbb{C}[\widehat{Gr}_{m,n}]$.

Next: Explain "cluster adjacency" for amplituhedron.
Fix k, n, m and $Z \in Mat^{>0}_{n,k+m}$ as usual. Have $\tilde{Z}: Gr^{\geq 0}_{kn} \to Gr_{k,k+m}$

Notation: If S_π is a cell of $Gr^{\geq 0}_{k,n}$, write $Z^\circ_\pi = \tilde{Z}(S_\pi)$ and $Z_\pi = \overline{\tilde{Z}(S_\pi)}$

Recall: If $\tilde{Z}$ is injective on a km-dim'l cell S_π, call Z_π a *tile*.

Def: Let Z_π be a tile. We say Z_τ is a *facet* of Z_π if:
- $Z_\tau \subset \partial Z_\pi$
- cell $S_\tau \subseteq \overline{S_\pi}$
- Z_τ has codim 1 in Z_π.

Cluster adjacency for $m=2$ ampl $\mathcal{A}_{n,k,2}(Z)$ was first conjectured by Lukowski–Parisi–Spradlin–Volovich:

Theorem (Parisi-ShermanBennett-W): Let Z_π be a tile of $A_{n,k,2}(Z)$. Then each facet of Z_π lies on a hypersurface $\langle Y Z_i Z_j \rangle = 0$, and the collection $\mathcal{C} = \{p_{ij}\}$ of Plucker coords corresp to facets is a collection of compat cluster variables for $Gr_{2,n}$.

twistor coord

(ie. the set of arcs $\{ij\}$ is noncrossing in the n-gon)

Moreover, if p_{hl} is compatible w/ $\mathcal{C}$, then $\langle Y Z_h Z_l \rangle$ has a fixed sign on Z_π°.

Cluster Adjacency Conj: (P-SB-W) Let Z_π be a tile of $A_{n,k,m}(Z)$. (maybe want $m=2$). Then each facet Z_T of Z_π lies on a hypersurface cut out by a functionary $F_T(\langle Y Z_I \rangle)$, & if we identify F_T w/ a polynomial in Plucker coordinates, we get a collection $\{F_T\}$ of compatible cluster variables in $\mathbb{C}[\widehat{Gr_{m,n}}]$.

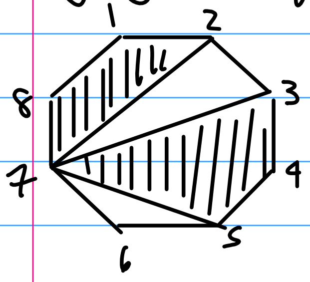
Theorem: (EvenZohar-Lakrec-Tessler-P-SB-W)

The cluster adj conj is true for the BCFW tiles in the $m=4$ amplituhedron.

Why is $m=2$ Thm true? First, we can classify the tiles.

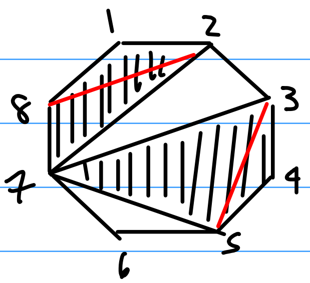
Theorem (P-SB-W, conj by L-P-S-V):

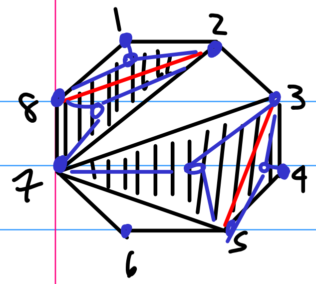
The positroid tiles for $A_{n,k,2}(Z)$ ⟷ collections of nonoverlapping grey polygons in n-gon w/ total "area" k.



To construct the cell S_π:

- Choose triangulation of grey polygons into k triangles
- Put white vertex in each grey polygon, connected to 3 vertices





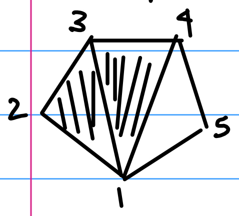
- Elemts of S_T are the $k \times n$ Kasteleyn matrices w/ rows columns indexed by white + black vertices

$$\begin{array}{c|cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & * & * & & & & & & * \\ 2 & & * & & & & & * & * \\ 3 & & & * & & * & & * & \\ 4 & & & * & * & * & & & \end{array}$$

Thm: There is way to associate a sign (+ or −) to each $*$ s.t. corresp $k \times n$ matrix $\in Gr_{kn}^{\geq 0}$.

Moreover, if each $*$ ranges over $\mathbb{R}_{>0}$ (or $\mathbb{R}_{<0}$ as needed), we get all elemts of a $(2k)$-dim'l cell in $Gr_{kn}^{\geq 0}$, on which $\tilde{Z}$ is injective.

Example: $k=2, n=5$

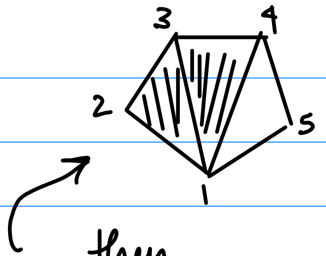


$$\rightsquigarrow \quad C = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \left(\begin{array}{ccccc} * & * & * & 0 & 0 \\ * & 0 & * & * & 0 \end{array} \right) \end{array} = \begin{pmatrix} c_1 & c_2 & c_3 & 0 & 0 \\ -c_4 & 0 & c_5 & c_6 & 0 \end{pmatrix}$$

$\in Gr_{25}^{\geq 0}$ when all $c_i \in \mathbb{R}_{>0}$

Above thm says such elements form a $\underline{\text{cell}}$ S_π of $Gr_{25}^{\geq 0}$

We can also describe the tile assoc to using twistor coord's:



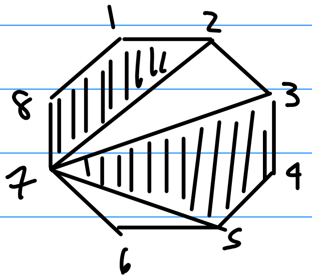
If Z_T denotes the tile associated to then the twistor coord's have following signs on $Y \in Z_T^\circ$:
For $i<j$, $\langle Y Z_i Z_j \rangle$ has sign $(-1)^{\text{area of grey region to left of arc } i \to j}$

Here: Positive twistor coord's: YZ_1Z_2, YZ_2Z_3, YZ_3Z_4, YZ_4Z_5
YZ_1Z_4, YZ_1Z_5

Neg: YZ_1Z_3, YZ_2Z_4

And finally the facets of tile correspond to border of grey polygons:

Eg:



There are 8 facets of this tile, cut out by the hypersurfaces
$\langle Y Z_7 Z_8 \rangle = 0$, $\langle Y Z_1 Z_8 \rangle = 0$, $\langle Y Z_1 Z_2 \rangle = 0$, $\langle Y Z_2 Z_7 \rangle = 0$,
$\langle Y Z_3 Z_4 \rangle = 0$, $\langle Y Z_4 Z_5 \rangle = 0$, $\langle Y Z_5 Z_7 \rangle = 0$, $\langle Y Z_3 Z_7 \rangle = 0$.

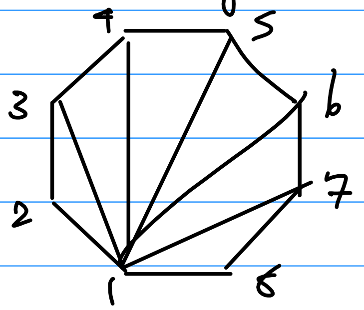
Clearly when we identify these w/ Plucker coord's, we get collection of compat clust variables. And all clust var's compat w/ these "facet variables" have fixed sign on the tile.

Q: Now that we understand tiles, when do they fit together to tile the amplituhedron?

Thm (Bao-He):

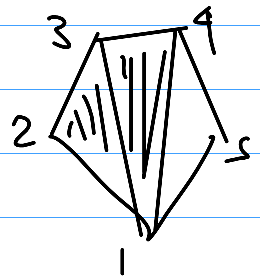
$\mathcal{A}_{n,k,2}(Z) = \bigcup_{T}$ tiles Z_T

where T ranges over all tiles



where k of these triangles are colored grey.

eg $\mathcal{A}_{5,2,2} =$



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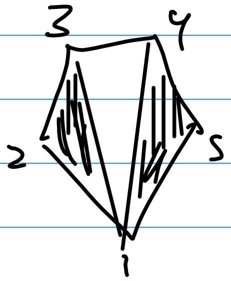

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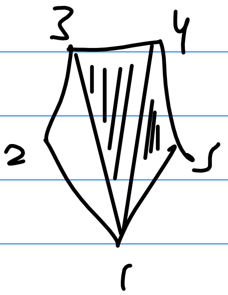


Moreover the open tiles Z_T° are pairwise disjoint, ie. $\{Z_T\}$ is a tiling.

Open: Classify all tilings of $\mathcal{A}_{n,k,2}$.

(We do have one classification:
(P-SB-W) Tilings of $\mathcal{A}_{n,k,2} \Leftrightarrow$ finest tilings of $\Delta_{k+1,2}$ by positroid polytopes)

Can use our description of tiles + Bao-He result to give char. of $m=2$ amplituhedron:

(Conj by AH-Thomas) ($\subseteq$ proved by AH-Thomas & Karp-W)

Thm: (P-SB-W) The $m=2$ amplituhedron is

$$\left\{ Y \in Gr_{k,k+2} \;\middle|\; \langle Y Z_i Z_{i+1} \rangle > 0 \text{ and the sequence } \langle YZ_1Z_2\rangle, \langle YZ_1Z_3\rangle, \dots, \langle YZ_1Z_n\rangle \text{ has } k \text{ sign flips} \right\}$$

To understand thm, need lemma which expresses twistor coords of CZ in terms of Pluckers of C and of Z:

Lemma: For $C \in Gr_{kn}$ and $Z \in Mat^{>0}_{n,k+m}$ we have

$$\langle CZ, Z_{i_1}, \dots, Z_{i_m} \rangle = \sum_{J=\{j_1<\dots<j_k\} \in \binom{[n]}{k}} p_J(C) \langle Z_{j_1}, \dots, Z_{j_k}, Z_{i_1}, \dots, Z_{i_m} \rangle.$$

So eg for $m=2$, we have

$$\langle CZ, Z_i, Z_h \rangle = \sum_{J \in \binom{[n]}{k}} p_J(C) \underbrace{\langle Z_{j_1} \dots Z_{j_k}, Z_i, Z_h \rangle}$$

(Note how our previous computation were special cases of this)

To figure out sign of need to do swaps until the subscripts are increasing.

Note that if $h=i+1$, $\langle CZ, Z_i, Z_{i+1} \rangle$ is pos!

The statement about sign flips follows from the Bao-He Thm plus our characterization of tiles.

Lecture 3:

Recall: Fix n, k, m w/ $k+m \leq n$.

Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ be $n \times (k+m)$ matrix w/ max'l minors pos.

$$\begin{array}{cc} & k+m \\ n & \begin{pmatrix} \qquad \\ \\ \end{pmatrix} \end{array}$$

Let $\tilde{Z} : Gr_{kn}^{\geq 0} \longrightarrow Gr_{k,k+m}$ be map

$$ {}_k\Big(\overset{n}{C} \Big) \mapsto {}_k\Big(\overset{n}{C} \Big) {}_n\Big(\overset{k+m}{Z} \Big) = {}_k\Big(\overset{k+m}{CZ} \Big)$$

(where identify matrices w/ row spans)

Let $\mathcal{A}_{n,k,m}(Z) := \tilde{Z}(Gr_{kn}^{\geq 0}) \subseteq Gr_{k,k+m}$

Def: (AH-Thomas) Fix Z as above, & let $Y \in Gr_{k,k+m}$ (thought of as matrix).

Write rows of Z as $\left(\begin{array}{c} Z_1 \\ \hline \vdots \\ \hline Z_n \end{array}\right)$.

Given $I = \{i_1 < \dots < i_m\} \subset [n]$, define the twistor coordinate

$$\langle Y Z_I \rangle = \langle Y Z_{i_1} \dots Z_{i_m} \rangle := \det \left(\begin{array}{c} Y \\ \hline Z_{i_1} \\ \hline \vdots \\ \hline Z_{i_m} \end{array}\right)$$

Cluster Adjacency Conj: (P-SB-W) Let Z_π be a tile of $\mathcal{A}_{n,k,m}(Z)$. (maybe want $m=2$). Then each facet Z_T of Z_π lies on a hypersurface cut out by a functionary $F_T(\langle Y Z_I \rangle)$, & if we identify F_T w/ a polynomial in Plucker coordinates, we get a collection $\{F_T\}$ of compatible cluster variables in $\mathbb{C}[\widehat{Gr_{m,n}}]$.

Today I'll discuss what we know for $m=4$...

For $m=2$ we had classification of all tiles.

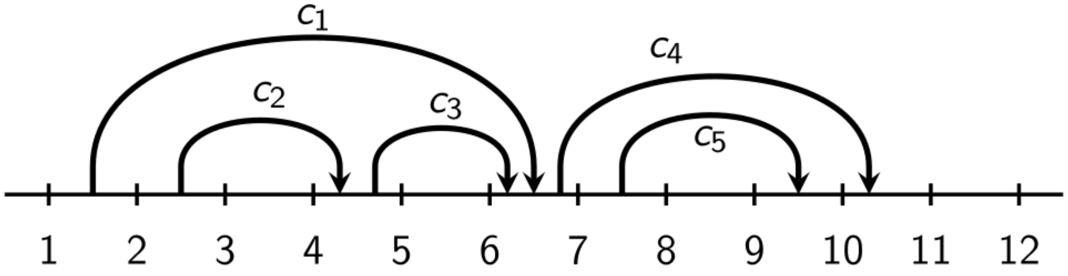
Open: Classify tiles of $\mathcal{A}_{n,k,4}(Z)$!

Open: Classify tilings of $\mathcal{A}_{n,k,4}(Z)$.

However, we know many tiles.

Thm (EvenZohar–Lakrec–Tessler):
One can associate a $4k$-dim'l cell S_D of $Gr_{k,n}^{\geq 0}$ to each chord diagram D on $[n]$ w/ k chords.
$\tilde{Z}$ is injective on S_D so $Z_D := \tilde{Z}(S_D)$ is a tile, & the collection $\{Z_D \mid D \text{ such a chord diagram}\}$ gives a tiling.

$n=12$
$k=5$



$$\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
\alpha_1 & \beta_1 & 0 & 0 & 0 & \gamma_1 & \delta_1 & 0 & 0 & 0 & 0 & \varepsilon_1 \\
\varepsilon_2\alpha_1 & \varepsilon_2\beta_1+\alpha_2 & \beta_2 & \gamma_2 & \delta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\varepsilon_3\alpha_1 & \varepsilon_3\beta_1 & 0 & \alpha_3 & \beta_3 & \gamma_3 & \delta_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_4 & \beta_4 & 0 & 0 & \gamma_4 & \delta_4 & \varepsilon_4 \\
0 & 0 & 0 & 0 & 0 & \varepsilon_5\alpha_4 & \varepsilon_5\beta_4+\alpha_5 & \beta_5 & \gamma_5 & \delta_5 & 0 & 0
\end{array}$$

↖ Matrix representing a $4k = 20$-dim'l cell of $Gr_{k,n}^{\geq 0} = Gr_{5,12}^{\geq 0}$.

There is a larger collection of BCFW cells (of which these are special case).

p.3

Thm (EZ-L-T — P-SB-W):

(1) $\tilde{Z}$ is injective on each BCFW cell S_r.

(2) The Cluster Adjacency Conj holds for all BCFW tiles.

(3) All collections of tiles produced by iterating the BCFW recurrence give a tiling of $A_{n,k,4}(Z)$.

Explain some ideas going into this + some facts we learn about cluster algebras for $Gr_{4,n}$.

Overall Ideas:

- BCFW cells can be built recursively, by (some simple operations like cyclic shift & reflection & insert zero) + taking the "BCFW product" $S_L \bowtie S_R$ of 2 smaller BCFW cells S_L and S_R
- We analyze facets of BCFW cells & their "vanishing functionaries"
- There is a homomorphism we call "promotion": For $1 \leq a < b \leq c \leq d \leq n$ with $b = a+1$, $d = n-1$, $c = n-2$,

$$\Psi : \mathbb{C}(Gr_{4,\{1,\dots,a,b,n\}}) \times \mathbb{C}(Gr_{4,\{b,\dots,c,d,n\}}) \to \mathbb{C}(Gr_{4,n})$$

(1) This map is a cluster quasihomomorphism, i.e. is compatible w/ the cluster structures on Gr's.

(2) If F is a functionary vanishing on a facet of $\tilde{Z}(S_L)$ or $\tilde{Z}(S_R)$, then $\Psi(F)$ vanishes on facet of $\tilde{Z}(S_L \bowtie S_R)$.

Applications to cluster algebras:

- prove that a large class of high-degree poly's in Plucker coords are clust. var's
- Build explicit new clusters + seeds for $Gr_{4,n}$. indexed by chord diagrams D

[These clusters contain the functionaries vanishing on facets of tile Z_D]

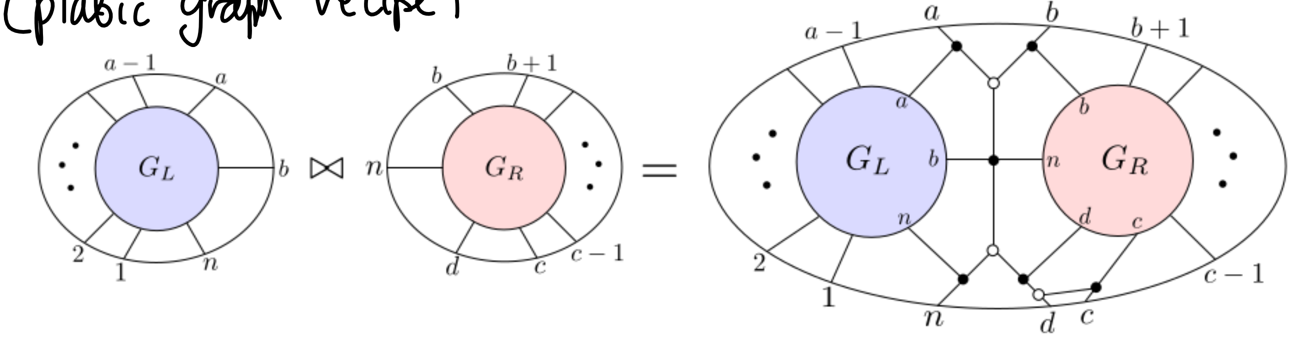
BCFW Product: Let $N_L = \{1,2,\ldots,a,b,n\}$ and $N_R = \{b,b+1,\ldots,n\}$

If S_L a cell of Gr_{k_L,N_L} and S_R cell of Gr_{k_R,N_R},

(s.t. $\{a,b,n\}$ coindep in S_L and $\{b,c,d,n\}$ coindep in S_R)

We define $S_L \bowtie S_R$ as follows:

(1) (plabic graph recipe)



(2) (matrix recipe)

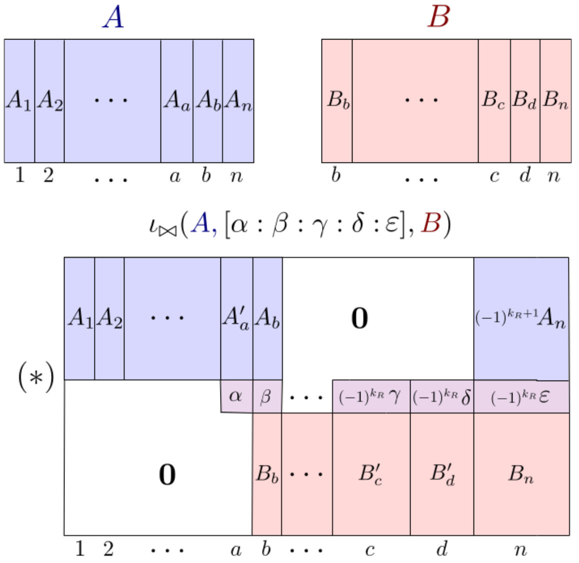


FIGURE 9. The image of $(A, [\alpha:\beta:\gamma:\delta:\varepsilon], B)$ under the BCFW map $\iota_{\bowtie}$. Here, $A'_a := A_a + \frac{\alpha}{\beta}A_b$, $B'_d := B_d + \frac{\delta}{\varepsilon}B_n$, $B'_c := B_c + \frac{\gamma}{\delta}B'_d$, and in a standard abuse of notation we identify a matrix with its rowspan.

p.5

Now we define BCFW cells recursively as follows:

Definition 7.1 (General BCFW cells). The set of *general BCFW cells* is defined recursively:

(Base case) For $k = 0$, the trivial cell $\mathrm{Gr}_{0,n}^{>0}$ is a general BCFW cell.

(Insert zero) If S is a general BCFW cell, then so is any cell obtained by inserting a zero column.

(Cyclic shift + reflect) If S is a general BCFW cell, then so is any cyclic shift or reflection of S.

(Product) If S_L and S_R are general BCFW cells on N_L and N_R as in Definition 6.4, then so is their BCFW product $S_L \bowtie S_R$.

Suppose we know the functionaries cutting out facets of $\tilde{Z}(S_L)$ and $\tilde{Z}(S_R)$.

To understand the functionaries cutting out the facets of $\tilde{Z}(S_L \bowtie S_R)$, we need product promotion:

notation for P_{bcdn}

$$\Psi_{ac} = \Psi : \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_L}) \times \mathbb{C}(\widehat{\mathrm{Gr}}_{4,N_R}) \to \mathbb{C}(\widehat{\mathrm{Gr}}_{4,n})$$

induced by the following substitution:

$$\text{(10)} \quad \text{on } \widehat{\mathrm{Gr}}_{4,N_L}\colon \quad b \mapsto b - \frac{\langle b\,c\,d\,n\rangle}{\langle a\,c\,d\,n\rangle} a \qquad = \frac{(ba)\cap(cdn)}{\langle a\,c\,d\,n\rangle}$$

$$\text{(11)} \quad \text{on } \widehat{\mathrm{Gr}}_{4,N_R}\colon \quad n \mapsto n - \frac{\langle a\,b\,c\,n\rangle}{\langle a\,b\,c\,d\rangle} d + \frac{\langle a\,b\,d\,n\rangle}{\langle a\,b\,c\,d\rangle} c = \frac{\langle a\,c\,d\,n\rangle}{\langle a\,b\,c\,d\rangle} b - \frac{\langle b\,c\,d\,n\rangle}{\langle a\,b\,c\,d\rangle} a \qquad = \frac{(ba)\cap(cdn)}{\langle a\,b\,c\,d\rangle}$$

$$\text{(12)} \quad \text{on } \widehat{\mathrm{Gr}}_{4,N_R}\colon \quad d \mapsto d - \frac{\langle a\,b\,d\,n\rangle}{\langle a\,b\,c\,n\rangle} c \qquad = \frac{(dc)\cap(abn)}{\langle a\,b\,c\,n\rangle}$$

Let $\mathcal{A}'$ be the cluster algebra obtained from $\mathbb{C}[\widehat{\mathrm{Gr}}_{4,n}]$ by freezing the additional clust variables $\langle abcd\rangle$, $\langle abcn\rangle$, $\langle acdn\rangle$, $\langle bcdn\rangle$. (+ 5 others)

Theorem 1: Ψ is a cluster quasi-h'ism from $\mathbb{C}[\mathrm{Gr}_{4,N_L}] \times \mathbb{C}[\mathrm{Gr}_{4,N_R}] \longrightarrow \mathcal{A}'$.

In particular, it maps cluster var's / cluster to c.v's/clusters. up to Laurent monomial in frozens.

Thm 2: Suppose $\tilde{Z}(S_L \bowtie S_R)$ is a BCFW tile.
Then the functionaries cutting out its facets are either:
(1) $\langle abcd\rangle, \langle abcn\rangle, \langle abdn\rangle, \langle acdn\rangle, \langle bcdn\rangle$
(2) $\Psi(F)$ where F a functionary cutting out facet of $\tilde{Z}(S_L)$ or $\tilde{Z}(S_R)$

Thms 1 and 2 allow us to prove Cluster Adjacency for BCFW tiles by induction.

Applications for cluster algebras

Notation for some quadratic poly's in Pluckers:

$$\langle abc \,|\, \underset{\curvearrowleft\,\curvearrowright}{de} \,|\, fgh\rangle := \langle abcd\rangle\langle efgh\rangle - \langle abce\rangle\langle dfgh\rangle$$

Some deg $(k+1)$ "chain poly's" in Pluckers:

$$\langle a_0 b_0 c_0 \,|\, d_{10} d_{11} \,|\, b_1 c_1 \,|\, d_{20} d_{21} \,|\, b_2 c_2 \,|\, \dots \,|\, d_{k0} d_{k1} \,|\, b_k c_k a_k\rangle$$

$$= \sum_{t\in\{0,1\}^k} (-1)^{t_1+\dots+t_k} \langle a_0 b_0 c_0 d_{1 t_1}\rangle\langle d_{1,1-t_1} b_1 c_1 d_{2,t_2}\rangle\langle d_{2,1-t_2} b_2 c_2 d_{3,t_3}\rangle \dots \langle d_{k,1-t_k} b_k c_k a_k\rangle$$

Cor: If $i<j<a<b<c<d<n$ then $\langle ijl \,|\, ba \,|\, cdn\rangle$ is a clust var

If $i<j<l<q<r<s<t<a<b<c<d<n$ then $\langle ijl \,|\, qr \,|\, st \,|\, ab \,|\, cdn\rangle$ is a clust var

Etc.