POLYLOGARITHMIC IDENTITIES IN CUBICAL HIGHER CHOW GROUPS

Essen and Bayreuth, January, 1998, Herbert Gangl[1] and Stefan Müller-Stach[1]

Abstract: *Bloch asked whether it is possible to obtain the five term relation for the dilogarithm in the context of the cubical version of his higher Chow groups, namely as the boundary of a 2-dimensional cycle in $\mathbb{A}^4$. We can give such a cycle and as a byproduct a realization of the isomorphism $B_2(F)_{\mathbb{Q}} \cong CH^2(F,3)_{\mathbb{Q}}$. We also shed some light on the relation between $B_3(F)_{\mathbb{Q}}$ and $CH^3(F,5)_{\mathbb{Q}}$ by proving further identities corresponding to functional equations of the trilogarithm, like the Kummer-Spence relation. Finally, we exhibit distribution relations in $CH^m(F,2m-1)_{\mathbb{Q}}$ for each m.*

I. Higher Chow Groups in a Nutshell

First let us give two different but equivalent definitions for Bloch's *higher Chow groups*:

(1) Simplicial version (cf. [1]): Here $\Delta^n = \mathbb{A}^n_F$ = affine n-space over an arbitrary field F with coordinates $t_0, ..., t_n$, $\sum t_i = 1$ and $\Sigma = \cup\{t_i = 0\} \subset \mathbb{A}^n$ is the union of all codimension one faces; a face is a subsimplex $\Delta^m \subset \Delta^n$ obtained by setting $n-m$ coordinates equal to zero. Let $Z^p(X,n)$ be the free abelian group of algebraic cycles of codimension p on $X \times \Delta^n$, meeting all faces of all codimensions properly. These groups form a simplicial abelian group:

$$...Z^r(X,3) \begin{smallmatrix}\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow\end{smallmatrix} Z^r(X,2) \begin{smallmatrix}\rightarrow\\\rightarrow\\\rightarrow\end{smallmatrix} Z^r(X,1) \begin{smallmatrix}\rightarrow\\\rightarrow\end{smallmatrix} Z^r(X,0)$$

and $CH^p(X,n)$ is defined to be the n-th homotopy group of this simplicial group. In other words, via the Dold-Kan theorem, $CH^p(X,n)$ is the n-th homology group of the complex

$$...Z^p(X,n+1) \xrightarrow{\partial} Z^p(X,n) \xrightarrow{\partial} Z^p(X,n-1) \xrightarrow{\partial} ... \xrightarrow{\partial} Z^p(X,0)$$

where $\partial = \sum (-1)^i \partial_i$ is given by the alternating sum of restrictions to faces.

(2) Cubical version (cf. [10]): Here instead $\mathbb{A}^n = (\mathbb{P}^1 \setminus \{1\})^n$ and the faces are defined by $x_i = 0, \infty$, while the boundary is given by $\partial = \sum (-1)^i (\partial_i^0 - \partial_i^\infty)$. $Z^p(X,n)$ is defined as above, except that one takes only the subcomplex of non-degenerate cycles. In [10] it is shown that this complex is quasiisomorphic to the simplicial version.

Higher Chow groups satisfy several formal properties like the homotopy axiom and localization. They also admit products and regulator maps to Deligne cohomology and étale cohomology. Their most important property is given by the theorem of Bloch [1] (refined by Levine in [11]) which puts them into relation with (the weight-graded pieces of) Quillen K-theory in the case of a smooth, quasiprojective variety X of dimension d over a field F:

$$K_n(X)^{(p)} \otimes \mathbb{Z}[\frac{1}{(n+d-1)!}] \cong CH^p(X,n) \otimes \mathbb{Z}[\frac{1}{(n+d-1)!}]$$

Let us mention a few further results:

Theorem([13],[16]). $K_n^M(F) \cong CH^n(F,n)$.

[1] Supported by the Deutsche Forschungsgemeinschaft

Theorem([14]). $CH^2(F,3) = K_3^{ind}(F) = K_3(F)/K_3^M(F)$.

Modulo torsion both sides are isomorphic to the Bloch group $B_2(F)$.

II. Computations in $CH^m(F, 2m-1)$.

We work in cubical coordinates. Let $G = G_n$ be the wreath product of the symmetric group S_n and $(\mathbb{Z}/2\mathbb{Z})^n$. It acts on $Z^p(F,n)\otimes\mathbb{Q}$ via permutation and inversion of coordinates. Define $\mathrm{sgn} : S_n \to \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{sgn}_j : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ to be the non-trivial 1-dimensional characters and

$$\chi = \mathrm{sgn}\cdot\prod_{j=1}^{n}\mathrm{sgn}_j\,,$$

then there is a natural choice of an idempotent

$$Alt_n : Z^p(F,n)\otimes\mathbb{Q} \to Z^p(F,n)\otimes\mathbb{Q}\,, \qquad Z \mapsto \frac{1}{|G|}\sum_{g\in G}\chi(g)\,g(Z)\,,$$

Note that we abbreviate Alt_n and sometimes Alt (when the context is clear) for the alternation over the full group G. Define $C^p(F,n)\subset Z^p(F,n)\otimes\mathbb{Q}$ to be the image of Alt together with the differential induced by ∂. The resulting homological complex

$$C^m(F,\cdot): \qquad \ldots \to C^m(F,2m) \to C^m(F,2m-1) \to \ldots \to C^m(F,m) \to 0$$

still computes $CH^m(F,n)_{\mathbb{Q}}$ by [10].

I. First let $m=2$: the complex $C^2(F,\cdot)$ has the acyclic subcomplex

$$\ldots \to C^1(F,1)\wedge C^1(F,3) \to C^1(F,1)\wedge C^1(F,2) \to C^1(F,1)\wedge \partial C^1(F,2) \to 0\,,$$

i.e. the truncation of the subcomplex consisting of subvarieties where one coordinate entry is constant. The proof of acyclicity is the same as in [12]. The quotient complex will be denoted by $A^2(F,\cdot)$. It is quasiisomorphic to $C^2(F,\cdot)$ and has certain advantages; for example cycles in $C^2(F,3)$ with one coordinate entry being constant have zero image in $A^2(F,3)$. We call such cycles *negligible*. Another advantage is—due to the fact that we mod out $C^2(F,2)$ by $C^1(F,1)\wedge\partial C^1(F,2)$—that the following diagram is well-defined and commutative:

$$\begin{tikzcd}
\mathbb{Z}[F-\{0,1\}] \arrow[rr, "\beta_2"] \arrow[d, "\rho_2"] & & \bigwedge^2 F^* \arrow[d, "\mathrm{id}\otimes\mathrm{id}"] \\
\dfrac{A^2(F,3)}{\partial A^2(F,4)} \arrow[rr, "\partial"] & & A^2(F,2)
\end{tikzcd} \tag{1}$$

where for a group G we denote by $\bigwedge^2 G$ the *subgroup* of $G\otimes G$ generated by $\{a\otimes b - b\otimes a \mid a,b\in G\}$.
The map β_2 is given on generators as $[a]\mapsto a\otimes(1-a)-(1-a)\otimes a$.
The map ρ_2 is defined as $\rho_2(a) = C_a \bmod \partial A^2(F,4)$ for $a\in F^*$, where C_a is defined in (2) below.

Notation: Given a rational map $\varphi : (\mathbb{P}^1)^d \to (\mathbb{P}^1)^n$, we let N_φ be the cycle associated to φ,

$$N_\varphi := \varphi_*\Big((\mathbb{P}^1)^d\Big)\cap\square^n\,,$$

i.e. the direct image in the sense of Fulton (1.4) [6].
For mnemonic reasons (cube=[ ]) we propose a "cubical" notation

$$[\varphi_1(x),\dots,\varphi_n(x)]\,,\qquad x=(x_1,\dots,x_n),$$

for $Alt_n(N_{(\varphi_1(x),\dots,\varphi_n(x))})$.
By abuse of notation, we will often identify φ with N_φ , and we introduce the further abbreviation for parametrized cycles of the special form associated to a rational function f

$$Z_f = Alt_3\big(N_{(x,1-x,f(x))}\big)\,.$$

Finally, we return to the definition of C_a (cf. [16]) as

$$C_a = Z_{\frac{x-a}{x}} = [x, 1-x, \frac{x-a}{x}]\,. \tag{2}$$

Note: In the following, equality signs between cycles are understood modulo negligible terms.

Lemma 1: *Let f, g, r and s be rational functions in one variable. Then there exists an element $W \in A^2(F,4)$ with*

$$\partial W = \Big[f(x), g(x), r(x)s(x)\Big] - [f(x), g(x), r(x)] - [f(x), g(x), s(x)]\,.$$

if all three cycles on the right hand side are admissible, i.e. intersect all faces in the correct dimension.
Proof. Let W be the parametrized cycle $W := [f(x), g(x), \frac{y-r(x)s(x)}{y-r(x)}, y]$. Then we compute (denoting the coordinates by $x_1,\dots,x_4$) that $W\cap\{x_1=0\}$, $W\cap\{x_1=\infty\}$, $W\cap\{x_2=0\}$ and $W\cap\{x_2=\infty\}$ are negligible, since one coordinate entry becomes constant. Next, $W\cap\{x_4=\infty\}=\emptyset$, since the third coordinate becomes 1. The remaining three boundaries give the three terms in the assertion. □

Corollary 1. (a) $C_a + C_{1-a} = C_1$ in $A^2(F,3)/\partial A^2(F,4)$.
(b) *Consider, for arbitrary $e, f \in F^\times$, the cycle $W(a,b,c,d) : x \mapsto [\frac{x-a}{e}, \frac{x-b}{f}, \frac{x-c}{x-d}]$ for pairwise distinct elements $a, b, c, d \in F$. Then*

$$W(a,b,c,d) = C_{\frac{c-a}{b-a}} - C_{\frac{d-a}{b-a}} \in A^2(F,3)/\partial A^2(F,4)$$

Furthermore,

$$[x, 1-\frac{x}{b}, 1-\frac{a}{x}] = C_{\frac{a}{b}} \in A^2(F,3)/\partial A^2(F,4)\,.$$

Proof: (a) We apply Lemma 1 with $f(x)=x$, $g(x)=1-x$, $r(x)=\frac{x-a}{x-1}$ and $s(x)=\frac{x-1}{x}$ to obtain $C_a = Z_{\frac{x-a}{x}} = Z_{\frac{x-a}{x-1}} + C_1$. Substituting $x \mapsto (1-x)$ and permuting the first two components in $Z_{\frac{x-a}{x-1}}$ yields

$$\Big[x, 1-x, \frac{x-a}{x-1}\Big] = \Big[1-x, x, \frac{1-x-a}{-x}\Big] = -\Big[x, 1-x, \frac{x-(1-a)}{x}\Big] = -C_{1-a}\,.$$

(b) We only prove the first assertion (the second can be shown in a similar way), and we restrict to the case $e=f=1$. The general case can be easily deduced by repeating some of the arguments used. $W(a,b,c,d) = W(0, b-a, c-a, d-a)$ by the transformation $x \mapsto x+a$. However

$$W(0, b-a, c-a, d-a) = [x, 1-\frac{x}{b-a}, \frac{x+a-c}{x+a-d}]$$

(here we multiplied with the admissible cycle $[x, \frac{1}{a-b}, \frac{x+a-c}{x+a-d}] \in C^1(F,1)\wedge C^1(F,2)$, which is allowed by lemma 1 in view of the definition of $A^2(F,\cdot)$) and therefore, using the transformation $x \mapsto (b-a)x$,

$$W(0, b-a, c-a, d-a) = [(b-a)x, 1-x, \frac{x-\frac{c-a}{b-a}}{x-\frac{d-a}{b-a}}]\,,$$

whence it ensues that it is equal (up to an admissible cycle in $C^1(F,1)\wedge C^1(F,2)$) to

$$[x,1-x,\frac{x-\frac{c-a}{b-a}}{x-\frac{d-a}{b-a}}]=[x,1-x,\frac{x-\frac{c-a}{b-a}}{x}]-[x,1-x,\frac{x-\frac{d-a}{b-a}}{x}]=C_{\frac{c-a}{b-a}}-C_{\frac{d-a}{b-a}}\,. \qquad \square$$

We proceed to describe the geometric nature of certain functional equations for the dilogarithm in the realm of higher Chow groups.

Theorem 1: (a) *If F contains a primitive n-th root of unity, then every $a\in F^*$ gives rise to a distribution relation:*

$$nC_{a^n}=n^2\sum_{\zeta^n=1}C_{\zeta a}\text{ in }A^2(F,3)/\partial A^2(F,4)$$

(b) *For $a\neq b,1-b$, and $a,b\neq 0,1$, one obtains the five term relation*

$$\mathcal{V}_{a,b}:=C_{\frac{a(1-b)}{b(1-a)}}-C_{\frac{1-b}{1-a}}+C_{1-b}-C_{\frac{a}{b}}+C_a=0\in A^2(F,3)/\partial A^2(F,4)\,.$$

(c) *The inversion relation holds:*

$$2(C_a+C_{\frac{1}{a}}-2C_1)=0\in A^2(F,3)/\partial A^2(F,4)\,.$$

(d) *Let $f:\mathbb{P}^1\to\mathbb{P}^1$ be any rational function of degree n. Then the following relation holds:*

$$2n^2C_{cr(f(x),a,b,c)}=2n\sum_{\gamma\in f^{-1}(c)}\sum_{\beta\in f^{-1}(b)}\sum_{\alpha\in f^{-1}(a)}C_{cr(x,\alpha,\beta,\gamma)}$$

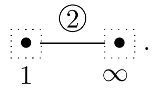
assuming that $x,a,b,c\in F$ and all α,β,γ are mutually distinct and lie in F.
Here $cr(a,b,c,d)$ denotes the cross ratio $\frac{(a-c)(b-d)}{(a-d)(b-c)}$.

Proof: (a) We give the proof for $n=2$: note that $2C_{a^2}=[t^2,1-t^2,\frac{t^2-a^2}{t^2}]$ (cf. [6], 1.4). Therefore by repeated application of Lemma 1 (check that in each step all cycles are defined, i.e. admissible):

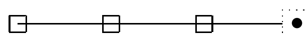
$$\begin{aligned}2C_{a^2}&=[t^2,1-t^2,\tfrac{t^2-a^2}{t^2}]=[t^2,1-t^2,\tfrac{t-a}{t}]+[t^2,1-t^2,\tfrac{t+a}{t}]\\&=[t^2,1-t,\tfrac{t-a}{t}]+[t^2,1+t,\tfrac{t-a}{t}]+[t^2,1-t,\tfrac{t+a}{t}]+[t^2,1+t,\tfrac{t+a}{t}]\\&=[t^2,1-t,\tfrac{t-a}{t}]+[t^2,1-t,\tfrac{t+a}{t}]+[t^2,1-t,\tfrac{t+a}{t}]+[t^2,1-t,\tfrac{t-a}{t}]\\&=4[t,1-t,\tfrac{t-a}{t}]+4[t,1-t,\tfrac{t+a}{t}]\\&=4C_a+4C_{-a}\,.\end{aligned}$$

For $n\geq 3$, the proof is similar, noting that $nC_{a^n}=[t^n,1-t^n,\frac{t^n-a^n}{t^n}]$ and $1-t^n=\prod(1-\zeta t)$ and $t^n-a^n=\prod(t-\zeta a)$.

(b) In table 1 a proof of the five term relation is given both in terms of cycles and (parallel to it) in terms of graphs; the latter visualize (and guide) the decomposition of cycles. One can play the following game: starting from a "distinguished position" (this corresponds to a sum of cycles $\sum C_{a_i}$), one is allowed to perform a number of "moves" (this corresponds to decomposing the cycles) of a certain kind which should eventually lead to another "distinguished position" (again, a sum $\sum C_{b_j}$). If the latter is different from the starting position, the difference $\sum[a_i]-\sum[b_j]$ gives a functional equation for the dilogarithm. The (undirected) graph encodes certain data of cycles whose coordinates are products of fractional linear transformations: we fix an ordering of the coordinates; the zeros of a coordinate are marked points which are connected via marked edges to its poles—also given as marked points. The mark underneath a point denotes the point viewed as lying in $\mathbb{P}^1$, the encircled number above an edge gives the number of the coordinate in the chosen ordering of the cycle coordinates. E.g. if the second coordinate is given by $1-x$, it is encoded as



The other information which is captured in the graph are the preimages of 1, these are pictured by a full-square with both the marking of the point (below) and the information about the coordinate number inside the square. In the above example we get 0 for the second coordinate, and this results in the picture $\underset{0}{\boxed{2}}$.

A dotted square in the example above means that none of the coordinates involved has image 1 at this point.
Fractional linear cycles are admissible if and only if in their corresponding graph any two adjacent edges are "glued" along a full-squared vertex (as opposed to a dotted-squared one). The latter condition corresponds in the language of cycles to boundary terms which have one coordinate being 0 or ∞ (and are a priori not admissible) but which are annihilated since another coordinate is equal to 1 and therefore vanishes.
A *distinguished position* is of the type , the information about coordinates is essentially redundant (but convenient for purposes of comparison as in the table).
A *move* is one of the following: (1) split up an edge into two, (2) move a full square from one vertex to another, and (3) the reverse operations to (1) and (2).
A move is *admissible* if both graphs before and after the move are admissible.

We could also introduce a direction of the graph, say, by substituting edges by arrows going from a zero to a pole and then we can give it a sign, depending on the permutation of the encircled marks and the direction of the arrows. This will determine a cycle unambiguously from its graph. Since we can get by without and we don't want to overload the picture, we omit it.
(c) follows from (b) and of corollary 1, (a):

$$\mathcal{V}_{a,b} + \mathcal{V}_{\frac{1}{a},\frac{1}{b}} - (C_b + C_{1-b} - C_1) - (C_{\frac{1}{b}} + C_{1-\frac{1}{b}} - C_1) = C_a + C_{\frac{1}{a}} - C_b - C_{\frac{1}{b}} - (C_{\frac{a}{b}} + C_{\frac{b}{a}} - 2C_1)\,.$$

Now add the corresponding expression where the roles of a and b are interchanged.

(d) For simplicity we restrict to the case of $f(x)$ being a polynomial of degree n and taking $a, b, c \in F$ such that all their preimages are distinct (i.e. they are non-critical points of the map f). Then for any α and β such that $f(\alpha) = a$, $f(\beta) = b$, the following decompositions hold:

$$\frac{f(x)-a}{b-a} = \frac{\prod_i (x-\alpha_i)}{\prod_i (\beta-\alpha_i)}, \qquad \frac{f(x)-b}{a-b} = \frac{\prod_j (x-\beta_j)}{\prod_j (\alpha-\beta_j)} \quad \text{and} \quad \frac{f(x)-c}{f(x)-a} = \frac{\prod_k (x-\gamma_k)}{\prod_l (x-\alpha_l)}\,.$$

Therefore reparametrizing nC_c, using $x \mapsto \frac{f(x)-a}{b-a}$, gives

$$nC_c = \Big[\prod_i \frac{(x-\alpha_i)}{(\beta-\alpha_i)}, \frac{f(x)-b}{a-b}, \frac{\prod_k (x-\gamma_k)}{\prod_l (x-\alpha_l)}\Big]$$

and we can decompose in the first coordinate in the following way

$$= \sum_i \Big[\frac{x-\alpha_i}{\beta-\alpha_i}, \frac{f(x)-b}{a-b}, \frac{\prod_k (x-\gamma_k)}{\prod_l (x-\alpha_l)}\Big]$$

since all the resulting cycles are admissible, and furthermore, we can decompose in the last coordinate which for convenience we write as a double product (which is matched up by a factor n on the left)

$$n^2 C_c = \sum_{i,k,l} \Big[\frac{x-\alpha_i}{\beta-\alpha_i}, \frac{f(x)-b}{a-b}, \frac{x-\gamma_k}{x-\alpha_l}\Big]\,.$$

Now we need to distinguish two cases $i = l$ and $i \neq l$. The latter one poses no problem, we can decompose in the second coordinate into terms $\frac{x-\beta_j}{\alpha-\beta_j}$, and a typical cycle

$$\left[\frac{x-\alpha_i}{\beta-\alpha_i}, \frac{x-\beta_j}{\alpha-\beta_j}, \frac{x-\gamma_k}{x-\alpha_l}\right]$$

is admissible and—via corollary 1(b)—equivalent to $C_{\frac{\gamma_k-\alpha_i}{\beta_j-\alpha_i}} - C_{\frac{\alpha_l-\alpha_i}{\beta_j-\alpha_i}}$.

For the terms with $i = l$ we only need to decompose the second coordinate into factors $\frac{x-\beta_j}{\alpha_i-\beta_j}$ with the *same* α_i $\big($above we could have chosen any $\alpha \in f^{-1}(a)\big)$ to make sure of the admissibility of the resulting cycles, and then we can again invoke corollary 1(b) to obtain $C_{\frac{\gamma_k-\alpha_i}{\beta_j-\alpha_i}}$. This gives us the claim with a polynomial f, since two terms $t(i,l) = C_{\frac{\alpha_l-\alpha_i}{\beta_j-\alpha_i}}$ and $t(l,i)$ are equal up to 2-torsion (note that $2(C_x + C_{\frac{x}{x-1}}) = 0$, which follows from (c) and corollary 1(a)). $\square$

Corollary 2. $C_1 = C_{-1} = 0 \in CH^2(F,3)_{\mathbb{Q}}$.

Proof: Use (a) with $n = 2$ and (c) with $a = -1$, respectively. The outcome is $4C_{-1} = 4C_1$ and $4C_{-1} = -2C_1$, yielding $6C_1 = 12C_{-1} = 0$. $\square$

Returning to diagram (1), we see that the map induced by ρ_2 on the kernels factors (up to tensoring with $\mathbb{Q}$) through the *Bloch group* $B_2(F)$ (which is defined [14] as the quotient of $\ker \beta_2$ by the subgroup generated by the five term relation), thereby realising the well-known isomorphism $B_2(F)_{\mathbb{Q}} \cong CH^2(F,3)_{\mathbb{Q}}$.

II. Now let $m \geq 3$. Consider the cycles of Bloch-Kriz ([2])

$$C_a = C_a^{(m)} : (x_1, ..., x_{m-1}) \mapsto \left[x_1, ..., x_{m-1}, 1-x_1, 1-\frac{x_2}{x_1}, ..., 1-\frac{x_{m-1}}{x_{m-2}}, 1-\frac{a}{x_{m-1}}\right]$$

in $C^m(F, 2m-1)$ (m will be clear from the context so we suppress the superscript).

As in the case $m = 2$ we modify the complex $C^m(F,\cdot)$ in such a way that we can manipulate parametrized cycles modulo negligible ones. To simplify our computations a little bit, we mod out by a subcomplex $\mathcal{J}^m(F,\cdot)$ which was already considered by Bloch and Kriz in [2]. It can be constructed as follows: look first at the graded algebra

$$\bigoplus_{p,n} C^p(F,n)$$

with multiplication given by the product in cubical coordinates (see [11]). The differential graded ideal generated by all $C^p(F,n)$ with $n \geq 2p+1$ and all $C^p(F,2p)$ for $p \geq 1$ is denoted by $\mathcal{J}^{\cdot}(F,\cdot)$. It contains for example all groups $\partial C^p(F,2p)$ for $p \geq 1$.

For $m = 3$, the group $\mathcal{J}^3(F,5)$ contains the subgroups

$$C^1(F,1) \wedge C^2(F,4), \qquad C^1(F,2) \wedge C^2(F,3) \quad \text{and} \qquad \partial C^3(F,6).$$

$\mathcal{J}^3(F,4)$ contains the subgroups

$$\partial C^1(F,2) \wedge C^2(F,3), \qquad C^1(F,1) \wedge \partial C^2(F,4) \quad \text{and} \qquad C^1(F,2) \wedge C^2(F,2).$$

The quotient complex $C^3(F,\cdot)/\mathcal{J}^3(F,\cdot)$ will be denoted by $A^3(F,\cdot)$. For number fields F, the complex $\mathcal{J}^3(F,\cdot)$ is known to be acyclic, so $A^3(F,\cdot)$ is quasiisomorphic to $C^3(F,\cdot)$ in this case. This is a consequence of Borel's theorem [4] by using the formulas for the regulator in [8].
In general, the acyclicity of $\mathcal{J}^3(F,\cdot)$ is not known and is related to the Beilinson-Soulé conjecture for $m = 2$ (cf. [2] and [15]).
In the following we will frequently omit terms which are contained in $\mathcal{J}^3(F,\cdot)$ and call them—as the corresponding ones in the case $m = 2$—*negligible*. As before, equality signs between cycles are understood modulo negligible terms.
We have again a commutative diagram

$$\begin{tikzcd} \mathbb{Z}[F-\{0,1\}] \arrow[r, "\beta_3"] \arrow[d, "\rho_3"] & \dfrac{\mathbb{Z}[F-\{0,1\}]}{R_2(F)} \wedge F^* \arrow[d, "\rho_2\wedge\mathrm{id}"] \\ \dfrac{A^3(F,5)}{\partial A^3(F,6)} \arrow[r, "\partial"] & A^3(F,4) \end{tikzcd} \tag{3}$$

Here $R_2(F)$ is defined as the subgroup generated by five term relations.
The map β_3 is defined on generators as $[a] \mapsto \big([a] \bmod R_2(F)\big) \wedge a$.
The map ρ_3 is defined as $[a] \mapsto C_a = C_a^{(3)}$.

The next lemma tells us how far the decomposition of certain cycles is from being bounded by some admissible cycle.

Lemma 4: *Consider a parametrized cycle of the form*

$$[f_1(x), f_2(y), f_3(x), f_4(x,y), f_5(y)]$$

where all f_i are rational functions and f_4 is a product of fractional linear transformations when considered as a function in the variable y.
(a) Assume that we can write $f_1(x) = g(x)h(x)$ for some rational functions g and h. Then the admissible cycle

$$W := [\frac{z-g(x)h(x)}{z-g(x)}, z, f_2(y), f_3(x), f_4(x,y), f_5(y)] \in A^3(F,6)$$

gives the following relation (provided all terms are admissible):

$$\partial(W) = [f_1(x), f_2, f_3, f_4, f_5] - [g(x), f_2, f_3, f_4, f_5] - [h(x), f_2, f_3, f_4, f_5]$$
$$+ \sum_{div(f_4)} \pm[\frac{z-g(x)h(x)}{z-g(x)}, z, f_2(y(x)), f_3(x), f_5(y(x))],$$

where we solve the equations $f_4(x,y) = 0$ (coefficient is $+1$) and $1/f_4(x,y) = 0$ (coefficient is -1) by the implicit function $y(x)$ in terms of x. Specifically, if e.g. $f_2(y(x)) = f_3(x)$, the corresponding term in the latter sum vanishes under the alternation.
A similar relation holds if we decompose f_i for $i \neq 4$ into two factors.
(b) *In the case $i = 4$, assume that we can write $f_4(x,y) = g(x,y)h(x,y)$ for some rational functions g and h. Then there is an admissible cycle $W \in A^3(F,6)$ such that*

$$\partial(W) = [f_1, f_2, f_3, f_4(x,y), f_5] - [f_1, f_2, f_3, g(x,y), f_5] - [f_1, f_2, f_3, h(x,y), f_5]\ ,$$

provided that all occurring terms are admissible.
(c) *Assume for each solution $y = r(x)$ of $f_4(x,y) = 0$ that $f_1(x) = f_2(r(x)) = g(x)h(x)$ and that either $g(r(x)) = g(x)$ or $g(r(x)) = h(x)$.*
Introduce a shorthand $Z(e_1, e_2) = [e_1(x), e_2(y), f_3(x), f_4(x,y), f_5(y)]$ for any two functions e_1, e_2 in one variable. Then

$$2Z(f_1,f_2) = Z\big(g,f_2\big) + Z\big(h,f_2\big) + Z\big(f_1,g\big) + Z\big(f_1,h\big)$$

and

$$2Z(f_1,f_2) = 2\big(Z(g,g) + Z(g,h) + Z(h,g) + Z(h,h)\big)\,.$$

Proof. (a) The first three boundaries $W \cap \{x_1 = 0\}$, $W \cap \{x_1 = \infty\}$ and $W \cap \{x_2 = 0\}$ produce the terms $[f_1(x), f_2, f_3, f_4, f_5] - [g(x), f_2, f_3, f_4, f_5] - [h(x), f_2, f_3, f_4, f_5]$. Then $W \cap \{x_2 = \infty\} = \emptyset$, since the first coordinate becomes 1. The next four boundaries $W \cap \{x_3 = 0\}$, $W \cap \{x_3 = \infty\}$, $W \cap \{x_4 = 0\}$ and $W \cap \{x_4 = \infty\}$ are in $\mathcal{J}^3(F,5)$ and therefore negligible. The boundaries $W \cap \{x_5 = 0\}$ and $W \cap \{x_5 = \infty\}$ create the sum $\sum_{div(f_4)} \pm [\frac{z-g(x)h(x)}{z-g(x)}, z, f_2(y(x)), f_3(x), f_5(y(x))]$. The last two boundaries $W \cap \{x_6 = 0\}$ and $W \cap \{x_6 = \infty\}$ are again negligible.
(b) is easier to prove than (a) and therefore left to the reader.
(c) The first equation follows by using the boundaries of the cycle

$$[\frac{z - f_1(x)}{z - g(x)}, z, f_2(y), f_3(x), f_4(x,y), f_5(y)] \;-\; [f_1(x), \frac{z - f_2(y)}{z - g(y)}, z, f_3(x), f_4(x,y), f_5(y)]\,,$$

and noting that the "deviation terms" (as given in (a)) from the decomposition are the same for both summands up to alternation. So they cancel in the difference and we are left with

$$Z\big(g,f_2\big) + Z\big(h,f_2\big) + Z\big(f_1,g\big) + Z\big(f_1,h\big) - 2Z(f_1,f_2) = 0\,.$$

For the second equation, one uses the first one and the boundaries of the cycles

$$[\frac{z - f_1(x)}{z - g(x)}, z, g(y), f_3(x), f_4(x,y), f_5(y)] \;+\; [\frac{z - f_1(x)}{z - g(x)}, z, h(y), f_3(x), f_4(x,y), f_5(y)] \qquad \text{and}$$

$$[g(x), \frac{z - f_2(y)}{z - g(y)}, z, f_3(x), f_4(x,y), f_5(y)] + [h(x), \frac{z - f_2(y)}{z - g(y)}, z, f_3(x), f_4(x,y), f_5(y)]\,.$$

where again all the "deviation terms" cancel under the alternation. □

Theorem 2: (a) *The Kummer-Spence relation for the trilogarithm*

$$\mathcal{S}(a,b) := C_{\frac{b(1-b)}{a(1-a)}} + C_{\frac{a(1-b)}{b(1-a)}} + C_{\frac{(1-a)(1-b)}{ab}} - 2C_{\frac{b}{a}} - 2C_{\frac{1-b}{a}} - 2C_{\frac{b}{1-a}} - 2C_{\frac{1-b}{1-a}} + 2C_{\frac{1}{1-a}} + 2C_{\frac{1}{a}} - 2C_{1-\frac{1}{b}}$$

holds in the form $4\,\mathcal{S}(a,b) = 0$ in $A^3(F,5)/\partial A^3(F,6)$ *whenever $a \neq b, 1-b$, and $a, b \neq 0, 1$.*
(b) *The inversion relation and the "3-term relation" hold*

$$4(C_a - C_{\frac{1}{a}}) = 0 \qquad\qquad 4(\mathcal{T}_a - \mathcal{T}_b) = 0\,, \quad \text{for} \quad \mathcal{T}_a = C_a + C_{\frac{1}{1-a}} + C_{1-\frac{1}{a}}\,.$$

Proof: (a) All cycles involved will be admissible.
If we use a phrase like "decompose the ith coordinate uv of a cycle into u and v", it is understood that there is an admissible cycle whose boundary is just the difference of the given cycle and the (sum of the) resulting ones after formally replacing the ith coordinate by u and v.

Let $\varphi_1(x) = \frac{x(1-x)}{a(1-a)}$, $\varphi_2(x) = \frac{a(1-x)}{x(1-a)}$ and $\varphi_3(x) = \frac{(1-a)(1-x)}{ax}$. Reparametrizing $C_{\varphi_1(b)}$ via $x \mapsto \varphi_1(x)$, $y \mapsto \varphi_1(y)$ produces the factor $(\deg \varphi_1)^2 = 4$ in the following equality

$$4C_{\varphi_1(b)} = [\varphi_1(x), \varphi_1(y), 1-\varphi_1(x), 1-\frac{\varphi_1(y)}{\varphi_1(x)}, 1-\frac{\varphi_1(b)}{\varphi_1(y)}]\,.$$

We apply lemma 4(c) twice to this cycle, first with $u(x) = x(1-x)$, thereby getting rid of the constants $a(1-a)$ in the first two coordinates, and then with $u(x) = x$, yielding four cycles all of which are the same after reparametrization, namely

$$[x, y, (1-\frac{x}{a})(1-\frac{x}{1-a}), (1-\frac{y}{x})(1-\frac{y}{1-x}), (1-\frac{b}{y})(1-\frac{b}{1-y})]\,.$$

Using lemma 4(b), we decompose the fourth coordinate into $1-\frac{y}{x}$ and $1-\frac{y}{1-x}$, leaving us with Z_1+Z_2, where

$$Z_1 = [x, y, (1-\tfrac{x}{a})(1-\tfrac{x}{1-a}), 1-\tfrac{y}{x}, (1-\tfrac{b}{y})(1-\tfrac{b}{1-y})] \quad \text{and}$$
$$Z_2 = [x, y, (1-\tfrac{x}{a})(1-\tfrac{x}{1-a}), 1-\tfrac{y}{1-x}, (1-\tfrac{b}{y})(1-\tfrac{b}{1-y})]\,.$$

Since we had four such cycles, we obtain $4C_{\varphi_1(b)} = 4(Z_1+Z_2)$.
We will now construct Z_2 in a different way, using $\varphi_2(b)$ and $\varphi_3(b)$, and then subtract the outcome from the one above. Consider

$$C_{\varphi_2(b)} = [\frac{a(1-x)}{x(1-a)}, \frac{a(1-y)}{y(1-a)}, \frac{x-a}{x(1-a)}, \frac{y-x}{y(1-x)}, \frac{b-y}{b(1-y)}]$$

and use, like above, lemma 4(c) with $u = a/(1-a)$ to get rid of the constants in the first two coordinates. Then we apply lemma 4(a) to decompose the third coordinate into $\frac{x-a}{1-a}$ and $\frac{1}{x}$. Note that the cycle with $\frac{1}{x}$ is just $C_{1-b^{-1}}$ (to be precise, we need lemma 4(c) again with $u=-1$).
The same procedure applies to $C_{\varphi_3(b)}$, the only difference is that the role of a and $1-a$ in the third coordinate are interchanged.
We again invoke lemma 4(a), noting that $f_1(x) = f_2(y(x))$ for $y(x) = x$, and we can "merge" the two remaining cycles in the third coordinate to

$$[\frac{1-x}{x}, \frac{1-y}{y}, \frac{x-a}{1-a}\cdot\frac{x-1+a}{a}, \frac{y-x}{y(1-x)}, \frac{b-y}{b(1-y)}]\,.$$

Successive decomposition of the fourth coordinate into $\frac{y-x}{1-x}$ and $\frac{1}{y}$ (the second one is negligible), and then of the fifth coordinate into $\frac{b-y}{1-y}$ and $\frac{1}{b}$ (again, the second one is negligible) leaves us with only one non-negligible cycle

$$[\frac{1-x}{x}, \frac{1-y}{y}, \frac{x-a}{1-a}\cdot\frac{x-1+a}{a}, \frac{y-x}{1-x}, \frac{b-y}{1-y}]\,,$$

which now finally can be decomposed in the first two coordinates via lemma 4(c) with $u(x) = 1-x$ into four terms T_i given (after an obvious reparametrization) by

$$T_1+T_2-T_3-T_4 :=$$
$$= [x, y, (1-\frac{x}{1-a})(1-\frac{x}{a}), 1-\frac{y}{x}, 1-\frac{1-b}{y}] + [x, y, (1-\frac{x}{1-a})(1-\frac{x}{a}), (1-\frac{y}{x})\frac{x}{x-1}, \frac{y-b}{y-1}]$$
$$-[x, y, (1-\frac{x}{1-a})(1-\frac{x}{a}), 1-\frac{y}{1-x}, 1-\frac{1-b}{y}] - [x, y, (1-\frac{x}{1-a})(1-\frac{x}{a}), (1-\frac{y}{1-x})\frac{x-1}{x}, \frac{y-b}{y-1}]\,.$$

Let us decompose both T_2 in the fourth coordinate into $1-\frac{y}{x}$ and $\frac{x}{x-1}$ (this will give a negligible cycle in $C^1(F,2)\wedge C^2(F,3)$) and then T_4, also in the fourth coordinate, into $1-\frac{y}{1-x}$ and $\frac{x}{x-1}$ (negligible), and call the resulting non-negligible terms T_2' and T_4', respectively.

T_1 and T_2' can be merged to Z_1, T_3 and T_4' can be merged to Z_2. So we have constructed Z_2 in a different way, as promised.

Summarizing, we can write $C_{\varphi_2(b)} + C_{\varphi_3(b)} \;=\; Z_1 - Z_2 + 2C_{1-b^{-1}}$, and with the above,

$$\sum_{j=1}^{3} C_{\varphi_j(b)} = 2Z_1 + 2C_{1-b^{-1}}\,.$$

at least if we multiply both expressions by 4, which gives the desired result using that Z_1 can be decomposed—by iterated application of lemma 4(a)—into cycles of the form C_α:

$$Z_1 = C_{\frac{1-b}{1-a}} + C_{\frac{1-b}{a}} + C_{\frac{b}{1-a}} + C_{\frac{b}{a}} - C_{\frac{1}{1-a}} - C_{\frac{1}{a}}\,.$$

(b) Let us introduce a shorthand $\langle x\rangle = C_x - C_{\frac{1}{x}}$, and $B = \frac{b-1}{b}$ and $A = \frac{a-1}{a}$, then

$$\mathcal{S}(a,b) - \mathcal{S}(a,1-b) = \Big\langle \frac{a(1-b)}{b(1-a)} \Big\rangle + \Big\langle \frac{(1-a)(1-b)}{ab} \Big\rangle + 2\Big\langle \frac{b}{b-1} \Big\rangle = \langle B/A\rangle + \langle BA\rangle + 2\langle 1/B\rangle \qquad \text{and}$$

$$\mathcal{S}(b,a) - \mathcal{S}(b,1-a) = \Big\langle \frac{b(1-a)}{a(1-b)} \Big\rangle + \Big\langle \frac{(1-b)(1-a)}{ba} \Big\rangle + 2\Big\langle \frac{a}{a-1} \Big\rangle\,.$$

Adding them up, we conclude

$$2\langle AB\rangle - 2\langle A\rangle - 2\langle B\rangle = 0\,.$$

Rewriting this gives $2\langle B/A\rangle + 2\langle BA\rangle - 2\langle B^2\rangle = 0$, and together with the above and the distribution relation for $n = 2$ we obtain $4\langle -B\rangle = 0$.
Similarly, we consider

$$2\big(\mathcal{S}(a,b) - \mathcal{S}(b,a)\big) = 2\Big(\Big\langle \frac{b(1-b)}{a(1-a)} \Big\rangle + \Big\langle \frac{a(1-b)}{b(1-a)} \Big\rangle\Big) \;+\; 4\mathcal{T}_a \;-\; 4\mathcal{T}_b$$

(we have left out terms of the form $4\langle x\rangle$ in the first equation), and using that the term in brackets on the right is equal to $\langle \frac{1-b}{1-a}\rangle$ by the above, we are done. $\square$

For the record, we mention also the validity of the distribution relations in general.

Proposition 8: *If F contains a primitive n-th root of unity, then every $a \in F$ gives rise to a distribution relation for any m:*

$$n^{m-1} C_{a^n} = n^{2m-2} \sum_{\zeta^n = 1} C_{\zeta a} \quad \text{in } A^m(F, 2m-1)/\partial A^m(F, 2m)\,.$$

Proof: Due to the special form of the cycle

$$\Big[x_1^n, \ldots, x_{m-1}^n, 1 - x_1^n, 1 - \frac{x_2^n}{x_1^n}, \ldots, 1 - \frac{a^n}{x_{m-1}^n}\Big]\,,$$

it can be shown that one can decompose the last m coordinates modulo $\partial A^m(F, 2m)$ into linear factors: the additional boundary terms which occur in lemma 4 do not appear since, for a typical solution $x_i = \zeta_n x_{i+1}$ of the corresponding equation, the resulting cycle vanishes under the alternation.
These decompositions are already sufficient to deduce the distribution relations along the lines of proposition 2 (a). $\square$

Concluding Remarks: Our computations for $m = 3$ could be important to study the relation between the third Bloch group $B_3(F) = \ker \beta_3 / R_3(F)$ and $CH^3(F,5)$. Here $R_3(F)$ is defined as the subgroup of $\mathbb{Z}[\mathbb{P}^1_F - \{0,1,\infty\}]$ generated by Goncharov's relation, the 3-term relation and the inversion relation (cf. [9], (5.17)).
Namely, if F is a field which has the property that $CH^3_{lin}(F,5)_{\mathbb{Q}}$ (in the sense of [7]) surjects onto $CH^3(F,5)_{\mathbb{Q}}$, then we conjecture that also the map $\rho_3 : B_3(F)_{\mathbb{Q}} \to CH^3(F,5)_{\mathbb{Q}}$ is surjective. This can probably be proved with our method of decomposing and merging fractional linear cycles.
If F is a field such that the map $CH^3_{lin}(F,5)_{\mathbb{Q}} \to CH^3(F,5)_{\mathbb{Q}}$ is even an isomorphism, then one can construct a well-defined inverse map to ρ_3 which is, on generators, defined by

$$\left[\frac{a_1^1 x + b_1^1 y + c_1^1}{a_2^1 x + b_2^1 y + c_2^1}, \ldots, \frac{a_1^5 x + b_1^5 y + c_1^5}{a_2^5 x + b_2^5 y + c_2^5}\right] \mapsto \sum_{i_1,\ldots,i_5=1}^{2} (-1)^{i_1+\ldots+i_5} \left(\begin{pmatrix} a^1_{i_1} \\ b^1_{i_1} \\ c^1_{i_1} \end{pmatrix}, \ldots, \begin{pmatrix} a^5_{i_5} \\ b^5_{i_5} \\ c^5_{i_5} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

as a linear combination of cross ratios of 6 points in $\mathbb{P}^2_F$ (cf. [8]) and thus giving rise to an element in $B_3(F)$. Therefore, if the map $CH^3_{lin}(F,5)_{\mathbb{Q}} \to CH^3(F,5)_{\mathbb{Q}}$ is bijective, then $CH^3(F,5)_Q \cong K_5^{(3)}(F)_{\mathbb{Q}}$ has a presentation in terms of generators and relations as conjectured by Zagier in the case of a number field.
For number fields a cubical version of the result of [7], together with the solution of the rank conjecture ([5]), imply that the map of Gerdes is at least surjective. Hence a more careful study of this map in terms of group homology has to be done in order to prove Zagier's conjecture.

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