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Regulators via iterated integrals (numerical computations)

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1. Motivation

Polylogarithms are known to give regulator values of elements in algebraic K-groups of number fields. In the case of the dilogarithm, Bloch found a criterion for elements in the free abelian group $\mathbb{Z}[F]$ for a number field F to produce such elements (cf. [**3**]), and for higher order polylogarithms an analogous criterion was proposed by Zagier which gave rise to his polylogarithm conjecture [**15**]. Beilinson and Deligne (cf. e.g. [**1**]), reinterpreted that criterion in terms of extension classes of mixed Tate motives over F, and realizations of the latter, given in terms of polylogarithms, provide real mixed Hodge–Tate structures; in a preprint [**2**] that unfortunately never made it into print they gave a proof of that reinterpretation, and a corresponding K-theoretic statement was independently shown by de Jeu (cf. [**10**]). As a consequence, given a natural number n, there are criteria for a formal linear combination $\sum_i \lambda_i[z_i]$ in $\mathbb{Z}[F]$ which guarantee that an appropriate single-valued version of the n-logarithm function (e.g. the function P_n in [**15**]) maps the image of $\sum_i \lambda_i[\sigma z_i]$ under a given embedding $\sigma : F \hookrightarrow \mathbb{C}$ to the regulator value of a suitable extension class.

Since polylogarithms can be expressed as iterated integrals, using a single 1-form of the kind $\frac{dt}{t-1}$ as well as further 1-forms of the type $\frac{dt}{t}$ only, one can ask whether more general iterated integrals also produce—possibly new—extension classes. Promising candidates are iterated integrals where we allow at least *two* 1-forms of the kind $\frac{dt}{t-1}$.

In his work on mixed Hodge structures and iterated integrals [**13**], Wojtkowiak generalizes the setup of the paper by Beilinson and Deligne [**1**] on the motivic interpretation of Zagier's conjecture to arbitrary iterated integrals involving only 1-forms with a linear form in the denominator. In this more general framework there arise new conditions on linear combinations in $\mathbb{Z}[F]$ (for a number field F) to represent an extension class in the category MTM$/F$ of mixed Tate motives over a field F (for the setting, see e.g. [**7**]), which then give rise to extensions of mixed Hodge-Tate structures after applying the associated iterated integral.

The aim of this note is to give examples representing such classes and having non-vanishing regulator values. For this we provide elements which satisfy the conditions mentioned above and evaluate them via some single-valued version for the associated iterated integrals. Finally we compare the result with the corresponding value of the Dedekind zeta function of F (the latter is motivated by Borel's theorem on regulators for the algebraic K-groups of F, combined with Zagier's polylogarithm conjecture). The output confirms numerically what the theory predicts, namely for the functions we consider (iterated integrals of type $\int \frac{dt}{t-1} \circ \frac{dt}{t} \circ \ldots \circ \frac{dt}{t} \circ \frac{dt}{t-1}$ of weight ≤ 5) we encounter the same regulator values (rationally) as for the classical polylogarithms, although for the most interesting case which we have investigated (the one of depth 5) the corresponding function is not expressible (cf. [**13**], §10.3) in terms of classical polylogarithms.

We want to emphasize that the tremendously useful software package GP-PARI [**9**] played an integral part for the experiments in this note.

2. Conditions to produce regulator values

2.1. Conditions from Zagier's polylogarithm conjecture. Let F be a number field, with r_1 real and $2r_2$ complex embeddings. Due to a famous result of A. Borel [**4**], we know that, using a suitable regulator map, its higher K-groups of odd order $K_{2n-1}F$ $(n \geq 2)$ can be mapped isomorphically, up to torsion, to a lattice of rank r_2 or $r_1 + r_2$, depending on whether n is even or odd; we will refer to such a lattice as a "higher regulator lattice". Bloch (unconditionally in the case of the dilogarithm, [**3**]) and Zagier (conjecturally for the higher cases, [**15**]) gave conditions for an element $\xi = \sum_i \lambda_i [z_i]$ in $\mathbb{Z}[F]$ to provide explicit entries in such a "higher regulator lattice" for F, at least up to a rational multiple. If those conditions are satisfied then any such entry takes the form $\mathcal{L}_{n,\sigma}(\xi) := \sum_i \lambda_i \mathcal{L}_n(\sigma\, z_i)$ for some embedding $\sigma : F \hookrightarrow \mathbb{C}$, where $\mathcal{L}_n(z)$ denotes a single-valued cousin (e.g. one can take the functions denoted by $\widetilde{\mathcal{D}}_n(z)$ or $P_n(z)$ in [**15**]) of the classical n-logarithm $Li_n(z) = \sum_{r\geq 1} z^r/r^n$, analytically continued to $\mathbb{C} \setminus \{0,1\}$ via an iterated integral of the form $-\int_0^z \frac{dt}{t-1} \circ \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{n-1 \text{ factors}}$.

For the dilogarithm the corresponding condition can be described using the second exterior power $\bigwedge^2 F^\times$ of the multiplicative group $F^\times$ of F: the condition for ξ alluded to above is simply to lie in $\ker(\beta_2)$ where the map $\beta_2 : \mathbb{Z}[F] \to \bigwedge^2 F^\times$ is given on generators as $[z] \mapsto z \wedge (1-z)$ (and $[0]$, $[1]$ are mapped to 0).

For the higher polylogarithms Zagier gave a similar "main condition", i.e. a good combination has to lie in $\ker(\beta_n)$, with $\beta_n : \mathbb{Z}[F] \to \bigotimes^{n-2} F^\times \otimes \bigwedge^2 F^\times$ which is defined on generators as $[z] \mapsto z^{\otimes(n-2)} \otimes z \wedge (1-z)$ for $n \geq 2$ (due to the symmetry of the situation we can replace $\bigotimes^{n-2}$ by Sym^{n-2}).

In addition to that main condition, though, he had to impose further "side conditions", coming from homomorphisms $\alpha_i : F^\times \to \mathbb{Z}$ ($i \in I$ for some index set I) and more generally from $\bigotimes^j F^\times$ to $\bigotimes^{j-1} F^\times$ $(1 \leq j \leq n-2)$ by applying these α_i to any one of the tensor factors on the left (we interpret $\bigotimes^0 F^\times$ on the right as $\mathbb{Z}$). By composing several of the α_i, one can map $\bigotimes^{n-2} F^\times$ to $\bigotimes^{k-2} F^\times$ for any $k = 2, \ldots, n-1$, and it turns out that the resulting (composed) induced homomorphisms $\alpha_{i_1} \circ \cdots \circ \alpha_{i_{n-k}} : \mathbb{Z}[F] \longrightarrow \mathbb{Z}[F]$ sending a generator $[z]$ to $\alpha_{i_1}(z) \cdots \alpha_{i_{n-k}}(z)\,[z]$, map elements from $\ker(\beta_n)$ to $\ker(\beta_k)$ for the corresponding k. Now the side conditions alluded to above amount to imposing that the image of ξ under any of those homomorphisms for any $k = 2, \ldots, n-1$ is not only in

$\ker(\beta_k)$, but moreover lies in a certain subgroup of "universal" elements coming from functional equations for $\mathcal{L}_k$ (some single-valued version of the k-logarithm as above). For many examples illustrating the above process we refer to [**15**] and [**6**].

The condition for an element $\xi \in \ker(\beta_n)$ to be a consequence of functional equations for the n-logarithm is very difficult to analyze (already for the simplest case of the dilogarithm there is no algorithm known for that). For this reason, Zagier has given a slightly different—and conjecturally equivalent—formulation where the above "lies in a certain subgroup of universal elements" is replaced by "vanishes under $\mathcal{L}_{k,\sigma}$ for all embeddings σ". This provides an effective check for conjectural triviality of Bloch elements.

Then one builds an inductive procedure: first we take linear combinations in $\ker(\beta_n)$ whose images in $\ker(\beta_2)$ under any composition of $n-2$ homomorphisms α_i as above vanish (numerically) when evaluated by the dilogarithm, then restrict to those linear combinations among them all of whose homomorphic images in $\ker(\beta_3)$ vanish (numerically) under the trilogarithm, and work our way up successively to $k = n-1$. Zagier's conjecture then implies that a combination ξ satisfying all those inductive conditions should map, up to multiplying by a rational number, to a vector $(\mathcal{L}_{n,\sigma})_\sigma$ inside the corresponding higher regulator lattice of F (cf. [**15**]).

In the framework of the paper by Beilinson and Deligne, the above conditions on ξ (in the non-numerical formulation) imply that it represents an extension class of mixed Tate motives in $\mathrm{Ext}^1_{\mathrm{MTM}/F}(\mathbb{Q}(0), \mathbb{Q}(n))$.

2.2. The conditions in the case $\Lambda_{10001}(z)$. In [**13**], Wojtkowiak has suggested a way to generalize the picture by invoking iterated integrals different from the ones for polylogarithms as candidates for regulator functions. We treat in this note (a single-valued version of) iterated integrals of the form

$$\Lambda_{\varepsilon_1,\dots,\varepsilon_n}(z) = \int_0^z \frac{dt}{t-\varepsilon_1} \circ \frac{dt}{t-\varepsilon_2} \circ \cdots \circ \frac{dt}{t-\varepsilon_n}\,,$$

where $\varepsilon_j \in \{0,1\}$ ($j = 1,\dots,n$), and in particular the case where $n = 5$ and $(\varepsilon_1,\dots,\varepsilon_5) = (1,0,0,0,1)$. The case ε_0 needs to be treated separately, cf. §3.2 below.

In this case, the "side conditions" on the corresponding linear combinations (in analogy to the above set-up) are somewhat more complicated, as there are now more different types of homomorphism for an element in the kernel $\ker(\tilde{\beta}_n)$ where

$$\tilde{\beta}_n : \mathbb{Z}[F] \;\longrightarrow \bigotimes\nolimits^{n-2} F^\times \otimes \bigwedge\nolimits^2 F^\times$$

is given on generators ($z \neq 0,1$) as

$$[z] \mapsto (1-z) \otimes z^{\otimes(n-3)} \otimes \big(z \wedge (1-z)\big)\,.$$

As usual, [0] and [1] are mapped to 0.

An example of a new type of homomorphism that we encounter here is obtained if we factor through (from tensors to antisymmetric tensors in the first two factors)

$$\bigwedge\nolimits^2 F^\times \otimes \bigotimes\nolimits^{n-4} F^\times \otimes \bigwedge\nolimits^2 F^\times\,,$$

where $\xi \in \ker(\tilde{\beta}_n)$ is mapped to an element in $\ker(\beta_2) \otimes (F^\times)^{\otimes(n-4)} \otimes \ker(\beta_2)$, which in turn is mapped homomorphically to $\mathbb{R}$, using the following function $\mathcal{L}_2 \otimes$

$\log|\cdot|^{\otimes(n-4)} \otimes \mathcal{L}_2$ (more precisely, we first need to apply individual embeddings $F \hookrightarrow \mathbb{C}$ for each tensor factor).

For the case in question the conditions for a $\xi = \sum \lambda_i [z_i] \in \mathbb{Z}[F]$ to provide a regulator value of an extension class in $\mathrm{Ext}^1_{\mathrm{MTM}/F}(\mathbb{Q}(0), \mathbb{Q}(5))$ via the single-valued version $D_{10001}(z)$ (defined in §3 below) attached to $\Lambda_{10001}(z)$ are given by Wojtkowiak ([**13**], in §10.3). We try to formulate his result in down-to-earth terms:

PROPOSITION 2.1. *Let F be a number field and let $\sum \lambda_i [z_i] \in \mathbb{Z}[F]$ satisfy the following conditions (M), (X1-3), (Y1-2):*

(M) *The main condition is*

$$\textstyle\sum_i \lambda_i\ (1-z_i) \wedge z_i \otimes z_i \otimes z_i \otimes (1-z_i) = 0 \qquad \text{in } \bigwedge^2 F^\times \otimes \left(F^\times\right)^{\otimes 3},$$

(X) *Conditions of the first kind. For any embedding $\sigma : F \hookrightarrow \mathbb{C}$ we have*

$$\begin{array}{ll} 1) & \sum_i \lambda_i\ \mathcal{L}_2(\sigma z_i) \otimes z_i \otimes z_i \otimes (1-z_i) = 0 \qquad \text{in } \mathbb{C} \otimes \left(F^\times\right)^{\otimes 3}, \\ 2) & \sum_i \lambda_i\ \mathcal{L}_3(\sigma z_i) \otimes z_i \otimes (1-z_i) = 0 \qquad \text{in } \mathbb{C} \otimes \left(F^\times\right)^{\otimes 2}, \\ 3) & \sum_i \lambda_i\ \mathcal{L}_4(\sigma z_i) \otimes (1-z_i) = 0 \qquad \text{in } \mathbb{C} \otimes F^\times . \end{array}$$

(Y) *Conditions of the second kind. For any embeddings $\sigma, \sigma' : F \hookrightarrow \mathbb{C}$ we have*

$$\begin{array}{ll} 1) & \sum_i \lambda_i \big(z_i \otimes \mathcal{L}_2(\sigma z_i) \mathcal{L}_2(\sigma' z_i)\big) = 0 \quad \text{in } F^\times \otimes \mathbb{R}\,. \\ 2) & \sum_i \lambda_i\ \mathcal{L}_3(\sigma z_i) \mathcal{L}_2(\sigma' z_i) = 0\,. \end{array}$$

Then the combination $\sum \lambda_i [z_i]$ gives an extension of $\mathbb{Q}$ by $(2\pi i)^5\mathbb{Q}$ in the category of mixed Tate motives over F.

We can view Proposition 2.1 as a generalization of Zagier's criteria (for elements in $\mathbb{Z}[F]$ representing elements in the algebraic K-theory of F which are mapped to a lattice under an appropriate single-valued polylogarithm function). In the spirit of Zagier's conjecture we now expect that the vectors $(\sum_i D_{10001}(\sigma\, z_i))_\sigma$ generate a full lattice in $\mathbb{R}^{r_1+r_2}$ when applied to elements satisfying the six criteria from that proposition.

Moreover, combining the above with Borel's Theorems (cf. [**4**]) we expect that the covolume of the (conjecturally) ensuing lattice is rationally, up to well-known factors, given by $\zeta_F(5)$.

CONJECTURE 2.2. *Let F be a number field of discriminant d_F. Then there are elements in $\mathbb{Z}[F]$, satisfying conditions (M), (X1–3) and (Y1–2) whose images under D_{10001} generate a lattice of full rank in $\mathbb{R}^{r_1+r_2}$, of covolume a rational number times $|d_F|^{9/2}\pi^{-5r_2}\zeta_F(5)$.*

The conditions above, with the exception of (Y2), can be rephrased in terms of homomorphisms, e.g. for (X1): $\sum_i \lambda_i\ \mathcal{L}_2(\sigma z_i)\, \alpha(z_i)\, \alpha'(z_i)\, \alpha''(1-z_i) = 0$ for all homomorphisms $\alpha, \alpha', \alpha'' : F^\times \to \mathbb{Z}$. Note that for simplicity we ignore torsion in $F^\times$ here (our computer program does in fact treat it, but in most of our example it is 2-torsion only, anyway). We will use the statement in this form for the description of the verification in §4 and in our examples in §5 below.

3. One-valued functions attached to $\Lambda_{10\ldots01}(z)$

3.1. The symbolic part of the calculation. In the notation of [**13**], the single-valued functions $D_{10^r1}(z)$ associated to $\Lambda_{10^r1}(z)$ are obtained using the Drinfeld associator $\Lambda_{\overrightarrow{01}}(z) := \sum_w c_w(z)w$ where w runs through all the words on the alphabet $\{X, Y\}$, and $c_w(z)$ is the corresponding iterated integral $\int_0^z$ over the composition of 1-forms of type $\frac{dt}{t}$ in place of X and $\frac{dt}{t-1}$ in place of Y. More precisely, $D_{10^r1}(z)$ is obtained as the coefficient of YX^rY in a power series associated to a particular automorphism of a certain Lie algebra (this Lie algebra, denoted $L(V)$ in [**13**], §8.0, with $V = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$, is obtained as a quotient of (a completion of) the free Lie algebra on two generators, and the automorphism mentioned arises from left multiplication by the above $\Lambda_{\overrightarrow{01}}(z)$, denoted $L_{\Lambda_{\overrightarrow{01}}(z)}$ in loc.cit., as $\frac{1}{2}\log(L_{\Lambda_{\overrightarrow{01}}(z)} \circ \overline{\overline{L}}^{-1}_{\Lambda_{\overrightarrow{01}}(z)})$); for details and notation, we refer to [**13**].

Specifically, $D_{10001}(z)$ is also denoted $\mathcal{D}^{f_5}(z, \overrightarrow{01})$ in [**13**] §10.3. For symbolic manipulations (which were performed in Mathematica) we can restrict ourselves to consider only the terms where at most two Y's and three X's appear.
Eventually, by taking appropriate real part $\Re$ or imaginary part $\Im$ and interpreting $\Lambda_0(z)$ as $\log(z)$, one finds the following one-valued function for $\Lambda_{101}(z)$:

$$\begin{aligned} D_{101}(z) = \;& \Re\Lambda_{101}(z) + \Im\Lambda_1(z)\Im\Lambda_{01}(z) - \Re\Lambda_{10}(z)\Re\Lambda_1(z) \\ & - \Im\Lambda_1(z)\Im\Lambda_0(z)\Re\Lambda_1(z) + \tfrac{1}{3}\Re\Lambda_1(z)\Re\Lambda_0(z)\Re\Lambda_1(z)\,. \end{aligned}$$

Similarly, the one for $\Lambda_{1001}(z)$ is given by:

$$\begin{aligned} D_{1001}(z) \;=\;& \Im\Lambda_{1001}(z) - \Im\Lambda_1(z)\Re\Lambda_{001}(z) - \Re\Lambda_{10}(z)\Im\Lambda_{01}(z) - \Im\Lambda_{100}(z)\Re\Lambda_1(z) \\ & -\Im\Lambda_1(z)\Im\Lambda_0(z)\Im\Lambda_{01}(z) + \frac{1}{3}\Re\Lambda_1(z)\Re\Lambda_0(z)\Im\Lambda_{01}(z) \\ & + \frac{1}{3}\Im\Lambda_{10}(z)\Re\Lambda_0(z)\Re\Lambda_1(z) + \Re\Lambda_{10}(z)\Im\Lambda_0(z)\Re\Lambda_1(z) \\ & - \frac{1}{3}\Re\Lambda_1(z)\Im\Lambda_0(z)\Re\Lambda_0(z)\Re\Lambda_1(z) + \frac{1}{6}\Im\Lambda_1(z)\Re\Lambda_0(z)\Re\Lambda_0(z)\Re\Lambda_1(z) \\ & + \frac{1}{2}\Im\Lambda_1(z)\Im\Lambda_0(z)\Im\Lambda_0(z)\Re\Lambda_1(z)\,. \end{aligned}$$

In fact, this function turns out to be identically zero (as had been predicted by Wojtkowiak).

In the above notation we have already tried to indicate some combinatorial structure of the terms involved. For any $r = 1, \ldots, n$ we partition the string $1\,0^{n-2}\,1 := 1\underbrace{0\cdots0}_{n-2}1$ into r substrings $B_1, \ldots, B_r$, which will be referred to as *blocks*, and we attach to each block B_j either the imaginary part or the real part of the associated functions $\Lambda_{B_j}(z)$. The blocks are separated by vertical bars, so e.g. the partition $1\,0|0|1$ has three blocks $1\,0$, 0 and 1. We introduce the following shorthand: we write $\overline{B}$ and B for $i \cdot \Im\Lambda_B(z)$ and $\Re\Lambda_B(z)$, respectively, and separate blocks by a $|$.

Then e.g. the function $D_{101}(z)$ above is represented as

$$[101] - [\overline{1}|\overline{01}] - [10|1] + [\overline{1}|\overline{0}|1] + \frac{1}{3}[1|0|1]\,.$$

So a priori we find 2^r terms for each partition, many of which come with a zero coefficient, though. We find in particular:

- the coefficient of any partition containing blocks of the form 0^k with $k > 1$ vanishes;
- if the number of "imaginary" blocks of a given partition and n have the same parity then the term has zero coefficient;
- if the final block in a partition does not have the form $\overline{0^{2k+1}1}$ or $0^{2k}1$ for some $k \geq 0$ (in the shorthand defined above), then the corresponding term has coefficient zero.

PROPOSITION 3.1. *In the above shorthand, the function $D_{10001}(z)$ is written as*

$$\begin{aligned}
&[10001] \;-\; [\bar{1}|\overline{0001}] \;-\; [10|001] \;-\; [\overline{100}|\overline{01}] \;-\; [1000|1] \\
&+[\bar{1}|\bar{0}|001] \;+\tfrac{1}{3}[1|0|001] \;+\; \tfrac{1}{3}[\overline{10}|0|\overline{01}] \;+\; [10|\bar{0}|\overline{01}] \;+\; [\overline{100}|\bar{0}|1] \;+\; \tfrac{1}{3}[100|0|1] \\
&-\tfrac{1}{3}[1|\bar{0}|0|\overline{01}] \;+\; \tfrac{1}{6}[\bar{1}|0|0|\overline{01}] \;-\; \tfrac{1}{2}[\bar{1}|\bar{0}|\bar{0}|\overline{01}] \\
&-\tfrac{1}{3}[\overline{10}|\bar{0}|0|1] \;-\; \tfrac{1}{2}[10|\bar{0}|\bar{0}|1] \;+\; \tfrac{1}{6}[10|0|0|1] \\
&+\tfrac{1}{6}[\bar{1}|\bar{0}|\bar{0}|\bar{0}|1] \;-\; \tfrac{1}{6}[\bar{1}|\bar{0}|0|0|1] \;+\; \tfrac{1}{6}[1|\bar{0}|\bar{0}|0|1] \;-\; \tfrac{7}{90}[1|0|0|0|1]\,.
\end{aligned}$$

More generally, we expect the following single-valued functions as the respective coefficient of $YX^{n-2}Y$ in the above power series (we have checked this symbolically up to $n = 12$):

$$\begin{aligned}
i^{\varepsilon}\, D_{10^{n-2}1}(z) \quad &= \quad \Re_n \Lambda_{10^{n-2}1}(z) - \\
-\sum_{r,s\geq 0}\sum_{1\leq b\leq n-1} (-1)^r \frac{\alpha_s}{r!}&\Big[\Re_{r+b+1-n}(1\,0^{n-r-s-b-1})|\underbrace{\bar{0}|\ldots|\bar{0}}_{r}|\underbrace{0|\ldots|0}_{s}|\Re_b\big(0^{b-1}1\big)\Big]
\end{aligned}$$

with α_s denoting the coefficient of x^s in the power series $\frac{x}{\sinh(x)} + \big(\frac{x}{\sinh(x)}\big)' = 1 - \frac{1}{3}x - \frac{1}{6}x^2 + \frac{7}{90}x^3 + \frac{7}{360}x^4 - \frac{31}{2520}x^5 \pm \ldots$, and where $\Re_j = \Re$ or $= i\,\Im$, depending on whether j is odd or even, and $\varepsilon = 0$ or 1 depending on whether n is odd or even (and the first block requires $n > r + s + b$, of course).

REMARK 3.2. *Somewhat different candidates for single-valued versions of the above functions (and many more) have in the meantime been given by F. Brown in* [**5**].

3.2. The computational aspect of Λ_{10001}. Note that $\Lambda_{\varepsilon_1\ldots\varepsilon_r}(z)$ ($\varepsilon_i \in \{0,1\}$, for $z \neq 0$) does not converge if $\varepsilon_1 = 0$. Therefore, in order to arrive at some computable (i.e. programmable) object, we treat $\Lambda_{0\ldots0}(z)$ as $\frac{1}{k!}\log^k(z)$ and produce the functions $\Lambda_{0\ldots01}$ from (the convergent) $\Lambda_{10\ldots0}$ via shuffle relations ("shuffle regularization"), and the latter ones are (up to sign) standard polylogarithms. At least inside the unit circle one has a rapidly convergent power series for computing $\Lambda_{10\ldots01}$, while outside the unit circle it can be given using an inversion relation. For the latter functions, such inversion relations have been provided by Wojtkowiak (cf. [**13**], §9 and [**14**], §10 (3)).
The main problems of evaluating the function arise close to the unit circle itself. A procedure given by Cohen, Lewin and Zagier [**6**] in the case of classical polylogarithms can be adapted to our situation, though: develop $\Lambda_{10\ldots01}(e^x)$ in a power series in x, of which most of the coefficients are expressed in terms of ζ-values (possibly evaluated at negative integers). The resulting expansion turns out to converge reasonably fast for an annulus $1/\rho < |x| < \rho$ for $\rho = 3$, say.

4. Description of the successive steps in the program

Let F be a number field of discriminant d_F and choose a set S of primes in $\mathbb{Q}$. We take a system $\mathcal{F}$ of fundamental S-units in F, i.e. a basis $\mathcal{F}_0$ for the S-units in $F^\times/\text{tors}$, together with a root ζ of 1 generating the torsion in $F^\times$, as provided by GP/PARI [**9**]. Note that for simplicity we ignore this torsion in the following (the actual program actually does respect the torsion, typically resulting in multiplying any linear combination by a factor of the order of ζ), hence we can disregard ζ for the following. Any S-unit has a unique representation in terms of $\mathcal{F}$. For each element $f_\nu \in \mathcal{F}_0$ we get a natural homomorphism $\alpha_\nu : \langle \mathcal{F} \rangle \to \mathbb{Z}$ picking the exponent of f_ν.

Moreover, we number the embeddings by first listing the pairs of complex conjugate embeddings $\sigma_1, \overline{\sigma}_1, \dots, \sigma_{r_2}, \overline{\sigma}_{r_2}$ and then appending successively the r_1 real embeddings. For the conditions involving $\mathcal{L}_2$ or $\mathcal{L}_4$ we only need to consider $\sigma_1, \dots, \sigma_{r_2}$, while for conditions involving $\mathcal{L}_3$ we invoke both real and complex embeddings $\sigma_1, \dots, \sigma_{r_1+r_2}$ (i.e. for each pair of complex embeddings we choose one).

4.1. Strategy for invoking the conditions. Initialize the procedure by searching for "many" exceptional S-units z_ν ($\nu \in \mathcal{V}$, an index set of size $N := |\mathcal{V}|$), i.e. S-units $z_\nu \in F^\times$ such that $1 - z_\nu$ is also an S-unit.

(M) For any $\nu \in \mathcal{V}$, represent z_ν and $1 - z_\nu$ in terms of $\mathcal{F}$, and associate to it the vector of integer entries arising from the different choices for the homomorphisms $\alpha_* : \langle \mathcal{F} \rangle \to \mathbb{Z}$ as above:

$$\alpha_i(z_\nu)\, \alpha_j(z_\nu)\, \alpha_k(1-z_\nu) \big[\alpha_l(z_\nu)\, \alpha_m(1-z_\nu) - \alpha_l(1-z_\nu)\, \alpha_m(z_\nu) \big] , \tag{1}$$

$1 \le i \le j \le s$, $1 \le k \le s$, $1 \le l < m \le s$. This provides a row m_ν of some integer matrix M_0 and the main obstruction (M) for giving an element $\sum_\nu \lambda_\nu [z_\nu]$ as in Proposition 2.1 is that the corresponding vector $(\lambda_\nu)_\nu$ has to lie in the kernel of M_0. Find the (integer) kernel I_0 of M_0, these form the first conditions on the linear combination of the rows (corresponding to the conditions on the z_ν).

(X1): Invoke further conditions using dilogarithmic conditions by computing the matrix M_2 of size $N \times \binom{s+1}{2}\binom{s}{2} s$

$$(M_2)_{\nu,\iota} = \big(\alpha_i(z_\nu)\, \alpha_j(z_\nu)\, \alpha_k(1-z_\nu)\, \mathcal{L}_2(\sigma_\ell z_\nu) \big)_{\nu,\iota} , \tag{2}$$

$1 \le i \le j \le s$, $1 \le k \le s$, $\ell = 1, \dots, r_2$, where $\sigma_\ell : F \hookrightarrow \mathbb{C}$. Numerically, the columns of the matrix $I_0 \cdot M_2^t$ span a lattice, of covolume a rational number times $|d_F|^{3/2} \pi^{-2(r_1+r_2)} \zeta_F(2)$, and we compute its integer kernel K_0. Then $I_1 := K_0 \cdot I_0$ will annihilate both M_0 and M_2.

(Y1): Then take the "products of dilogs"

$$(M_{2,2})_{\nu,k} = \big(\alpha_i(z_\nu)\, \mathcal{L}_2(\sigma_\ell z_\nu)\, \mathcal{L}_2(\sigma_m z_\nu) \big)_{\nu,k} , \tag{3}$$

$1 \le i \le s$, $1 \le \ell \le m \le r_2$, into a matrix $M_{2,2}$ (the index k runs through the $\binom{r_1+r_2}{2}$ pairs (ℓ, m)). Similarly, $I_1 \cdot M_{2,2}$ gives rise (numerically) to a lattice, more specifically of covolume a rational number times $\big(|d_F|^{3/2} \pi^{-2(r_1+r_2)} \zeta_F(2) \big)^2$, and we can find its (integer) kernel K_1. We define $I_2 := K_1 \cdot I_1$, which annihilates M_0, M_2 and $M_{2,2}$.

(X2): The next step consists in taking the trilogarithmic conditions

$$(M_3)_{\nu,\ell} = \big(\alpha_i(z_\nu)\,\alpha_j(1-z_\nu)\,\mathcal{L}_3(\sigma_\ell z_\nu)\big)_{\nu,\ell}\,, \tag{4}$$

$1 \le i,j \le s$, this time $\ell = 1,\ldots,r_1+r_2$, generating a matrix M_3, and we find the kernel K_2 of the matrix $I_2 \cdot M_3$ (whose rows give rise to a lattice of covolume a rational number times $|d_F|^{5/2}\pi^{-3r_2}\zeta_F(3)$) and then form $I_3 := K_2 \cdot I_2$, which annihilates M_0, M_2, $M_{2,2}$ and M_3.

(Y2): Compute further the expressions

$$(M_{3,2})_{\nu,k} = \big(\,\mathcal{L}_2(\sigma_\ell z_\nu)\,\mathcal{L}_3(\sigma_m z_\nu)\big)_{\nu,k}\,, \tag{5}$$

$1 \le \ell \le r_2$, $1 \le m \le r_1+r_2$ (the index k runs through the r_1r_2 pairs (ℓ,m)). The resulting matrix $M_{3,2}$ should give an integer kernel K_3 for the lattice generated by the columns of $I_3 \cdot M_{3,2}$. The matrix $I_4 := K_3 \cdot I_3$ annihilates M_0, M_2, $M_{2,2}$, M_3 and $M_{3,2}$.

(X3): As a final preliminary step, consider the expressions in

$$(M_4)_{\nu,\ell} = \big(\alpha_i(z_\nu)\,\mathcal{L}_4(\sigma_l z_\nu)\big)_{\nu,\ell}\,, \tag{6}$$

$1 \le i \le s$, $1 \le l \le r_2$, into a matrix M_4 and compute the integral kernel K_4 of $I_4 \cdot M_4$ and put $I_5 := K_4 \cdot I_4$ which annihilates all of the above as well as M_4.

(D_{10^31}) Now everything is in place to apply the function D_{10001}, namely we consider

$$(M_5)_{\nu,\ell} = \big(D_{10001}(\sigma_\ell z_\nu)\big)_{\nu,\ell}$$

$1 \le \ell \le r_1+r_2$, the entries of which form the matrix M_5.

$\zeta_F(5)$ We finally find that the columns of $R_5 := I_5 \cdot M_5$ generate (numerically) a lattice; moreover, we can determine its covolume and check whether it is a rational number (of small height) times

$$\frac{|d_F|^{9/2}}{\pi^{5r_2}} \cdot \zeta_F(5)\,.$$

In our examples, the corresponding ratio indeed looks rational, at least within the (40-digit) precision typically used.

5. A detailed example for a cubic field

Let $F = \mathbb{Q}(\theta)$ with $\theta^3 - \theta - 1 = 0$ ($d_F = -23$, signature $[r_1,r_2] = [1,1]$). Then θ is a generator for the group of units modulo torsion; the latter is generated by -1 and will be ignored in the following. We put $\mathcal{F}_0 = \{\theta\}$ and $\mathcal{F} = \langle\theta\rangle$.

Notation: In the following we indicate by $\doteq$ an "approximate equality", i.e. an equality which holds up to a given precision. Typically the computer performed the calculations up to 100 digits precision, except for the calculations for D_{10001} where we typically used 40 digits.

0. *Finding sufficiently many S-units.* The exceptional S-units z_ν that the computer found are the following twelve ones:

$$\begin{aligned}(z_\nu)_{\nu=1,\dots,12} \quad = \quad & \big(-\theta^2+2,\ -\theta+1,\ \theta^2-\theta,\ -\theta^2+\theta+1,\ \theta^2-1,\\ & -\theta^2+1,\ \theta,\ -\theta,\ \theta^2,\ \theta+1,\ -\theta^2-\theta,\ \theta^2+\theta+1\big)\,.\end{aligned}$$

We encounter the very special case (cf. Lewin's ladders as explained in [**15**], §9C) that there is essentially only one homomorphism $\langle\mathcal{F}\rangle \to \mathbb{Z}$ involved, namely the one with respect to the fundamental unit θ. The respective exponents are given as

$$\begin{aligned}\big(\alpha(z_\nu)\big)_\nu &= (-5,-4,-3,-2,-1,-1,1,1,2,3,4,5) \ \text{ and}\\ \big(\alpha(1-z_\nu)\big)_\nu &= (-1,1,-2,-3,-5,2,-4,3,-1,1,5,4).\end{aligned}$$

(M) This condition (the only one which is purely algebraic, as it does not involve any polylogarithms) is trivially satisfied, since $\mathcal{F}$ has rank 1, hence I_0 is the 12×12 identity matrix.

(X2) *Dilogarithm conditions.* The dilogarithm values are (only the non-real embedding σ_2 for each z_ν has to be considered since $r_2 = 1$, so we suppress it from the notation)

$$M_2 := \big(\alpha(1-z_\nu)\,\alpha(z_\nu)\,\alpha(z_\nu)\,\mathcal{L}_2(z_\nu)\big)_\nu \doteq$$

$$\doteq \begin{pmatrix}-11.78384203\\-7.541658902\\16.96873253\\-11.31248835\\2.356768406\\1.885414725\\-1.885414725\\-2.828122088\\3.770829451\\8.484366265\\37.70829451\\-47.13536814\end{pmatrix} \doteq 1.885414725 \begin{pmatrix}-6.250000000\\-4.000000000\\9.000000000\\-6.000000000\\1.250000000\\1.000000000\\-1.000000000\\-1.500000000\\2.000000000\\4.500000000\\20.00000000\\-25.00000000\end{pmatrix}$$

and the covolume $c_2 \doteq 1.885414725/4 \doteq 0.4713536814$ of the associated lattice is found to be[1]

$$c_2 \doteq \frac{3}{8}\,\frac{|d_F|^{3/2}}{\pi^{2(r_1+r_2)}}\,\zeta_F(2)\,.$$

We can find the integer kernel K_0 of M_2 and put $I_1 = K_0\cdot I_0 = K_0$. The latter is given in terms of the transpose of the following matrix I_1^t the entries of which are very close (within the given precision) to integers, therefore we round them off and are left with

[1]Note that we tend to display several zeros after the decimal point, which is more of a psychological feature (it reflects somewhat the satisfaction of the programmer verifying that numbers match in a computer calculation).

$$
I_1^t = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & -1 & 1 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & -1 \\
1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Then $I_1 \cdot M_2 \doteq 0$.

(Y1) *Products of dilogarithms.* After those preparations, we can expect condition (Y1) to give us a lattice from the matrix with entries products of dilogarithms

$$
M_{2,2} := \big(\alpha(z_\nu)\,\mathcal{L}_2(z_\nu)\,\mathcal{L}_2(z_\nu)\big)_\nu \doteq
$$

$$
\doteq \begin{pmatrix}
-1.110871464 \\ -0.8886971718 \\ -2.666091515 \\ -1.777394343 \\ -0.2221742929 \\ -0.8886971718 \\ 0.2221742929 \\ 0.8886971718 \\ 1.777394343 \\ 2.666091515 \\ 0.8886971718 \\ 1.110871464
\end{pmatrix}
\doteq -0.2221742929 \begin{pmatrix}
5.000000000 \\ 4.000000000 \\ 12.00000000 \\ 8.000000000 \\ 1.000000000 \\ 4.000000000 \\ -1.000000000 \\ -4.000000000 \\ -8.000000000 \\ -12.00000000 \\ -4.000000000 \\ -5.000000000
\end{pmatrix}
$$

with the covolume $c_{2,2} \doteq 0.2221742929$ being equal to c_2^2.

Note that $I_1 \cdot M_{2,2} \neq 0$, which shows that both conditions (X2) and (Y1) are needed. Instead, $I_1 \cdot M_{2,2}$ gives an 11×1-matrix with commensurable entries (obvious since $M_{2,2}$ already does), and one can give an integer kernel K_1 of it and multiply the result by I_1. Call the resulting matrix $I_2 = K_1 \cdot I_1$; it annihilates both M_2 and $M_{2,2}$. Its transpose is given by

$$I_2^t = \begin{pmatrix} 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 3 & 0 & -3 & 3 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 & 0 & 3 & 0 & 0 & -3 \\ 0 & -3 & 3 & 0 & 3 & 0 & 3 & -3 & 3 & 3 \\ 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 10 & -4 & -3 & 12 & -25 & 11 & -27 & 0 & 0 & 5 \\ 16 & -4 & -6 & 9 & -28 & 8 & -24 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 \\ 3 & -3 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(X2) *Trilogarithmic conditions.* The next step is to satisfy condition (X2). To this end, we compute the vector of trilogarithms multiplied by the corresponding homomorphisms (this time there are $r_1 + r_2 = 2$ embeddings σ_i of F into $\mathbb{C}$ to consider, which is reflected in the notation):

$$M_3 = \big(\alpha(1-z_\nu)\alpha(z_\nu)\,\mathcal{L}_3(\sigma_\ell\, z_\nu)\big)_{\nu,\ell} \doteq \begin{pmatrix} 4.034886241 & 4.677163445 \\ 3.081869446 & -3.987905178 \\ 6.013387652 & 2.498989698 \\ 6.553788692 & 0.2649913900 \\ 5.827735082 & -3.651760402 \\ 1.784944975 & -1.482786776 \\ -4.662188065 & 2.921408322 \\ -2.677417462 & 2.224180165 \\ -2.184596230 & -0.08833046334 \\ 3.006693826 & 1.249494849 \\ -15.40934723 & 19.93952589 \\ 16.13954496 & 18.70865378 \end{pmatrix}.$$

We find numerically that the rows of $I_2 \cdot M_3$ generate a lattice:

$$I_2 \cdot M_3 \doteq \begin{pmatrix} -21.09978266 & 10.54989133 \\ 2.549001106 & -6.082728165 \\ 4.230598524 & -5.120441520 \\ -9.198456008 & 45.46916271 \\ 44.28825961 & -16.13384529 \\ -2.476557456 & 1.839307179 \\ 35.98477232 & -7.173874034 \\ -12.23789998 & 0.7096939263 \\ 5.669220069 & -5.839752292 \\ -21.24466996 & 19.03673330 \end{pmatrix}.$$

The lattice property becomes more apparent if we multiply by the inverse matrix of the first 2×2-submatrix of the above and multiply by the common denominator 88, the result being:

$$\begin{pmatrix} 88.00000000 & -7.673865541E-104 \\ 0.E-124 & 88.00000000 \\ -11.00000000 & 55.00000000 \\ -52.00000000 & -748.0000000 \\ -198.0000000 & -110.0000000 \\ 9.000000000 & -11.00000000 \\ -174.0000000 & -198.0000000 \\ 63.00000000 & 99.00000000 \\ -17.00000000 & 55.00000000 \\ 70.00000000 & -154.0000000 \end{pmatrix}.$$

The determinant c_3 of the above $2{\times}2$-submatrix is $\doteq 101.452557625925282$ and it is expressed in terms of Dedekind zeta values via

$$c_3 \doteq \frac{11}{9} \cdot \frac{|d_F|^{5/2}}{\pi^{3r_2}} \zeta_F(3)\,.$$

Again, we can find some matrix K_2 which represents the integer kernel of the above, and $I_3 = K_2 \cdot I_2$ annihilates M_2, $M_{2,2}$ and M_3, where its transpose has the form

$$I_3^t = \begin{pmatrix} 2 & 0 & 1 & 3 & 0 & -1 & 0 & -3 \\ -1 & 5 & -1 & -2 & 3 & 0 & 0 & -3 \\ 0 & -1 & 2 & 0 & -2 & 2 & 1 & 0 \\ -3 & -1 & -2 & -4 & 5 & 2 & 0 & 3 \\ -2 & 0 & 3 & 1 & 0 & -3 & 0 & -1 \\ -7 & 3 & -2 & 7 & -10 & 3 & -2 & -7 \\ -8 & 0 & 4 & 4 & -7 & 4 & -4 & -4 \\ -2 & 2 & 2 & 0 & 0 & 0 & 4 & 0 \\ -1 & -3 & 1 & -1 & -1 & 4 & 2 & -2 \\ -2 & 2 & -1 & 1 & 0 & 1 & -2 & -1 \\ 0 & 1 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

(Y2) *Products of dilogarithm and trilogarithm.* I_3 does not annihilate the expressions arising from condition (Y2) which are displayed in the next equation

$$M_{3,2} = \big(\mathcal{L}_2(\sigma_1 z_\nu)\,\mathcal{L}_3(\sigma_\ell z_\nu)\big)_{\nu,\ell} \doteq \begin{pmatrix} 0.3803716968 & 0.4409196416 \\ 0.3631626273 & -0.4699284467 \\ -0.9448108025 & -0.3926359979 \\ 1.029717475 & 0.04163488907 \\ -0.5493848770 & 0.3442541418 \\ -0.8413403851 & 0.6989170060 \\ 0.5493848770 & -0.3442541418 \\ 0.8413403851 & -0.6989170060 \\ -1.029717475 & -0.04163488907 \\ 0.9448108025 & 0.3926359979 \\ -0.3631626273 & 0.4699284467 \\ -0.3803716968 & -0.4409196416 \end{pmatrix}$$

(note that for $\mathcal{L}_2$ we only use the first embedding) but it gives a lattice generated by the rows of

$$I_3 \cdot M_{3,2} \doteq \begin{pmatrix} -2.641083126 & -0.9458342228 \\ 5.505177483 & 0.08038099116 \\ -1.628466720 & -4.001815185 \\ -5.607617226 & 5.920075319 \\ 13.37917049 & -3.573318541 \\ -1.699781839 & -1.699781839 \\ -2.043364556 & -4.077663240 \\ 6.214401668 & -3.957091754 \end{pmatrix},$$

the span of which can be recognized with the naked eye using the same procedure as above (this time we multiply by 26):

$$\begin{pmatrix} 26.00000000 & 0.E-124 \\ -2.479835810E-105 & 26.00000000 \\ 114.0000000 & 47.00000000 \\ -172.0000000 & -109.0000000 \\ 108.0000000 & 115.0000000 \\ 48.00000000 & 15.00000000 \\ 116.0000000 & 46.00000000 \\ 116.0000000 & 85.00000000 \end{pmatrix}.$$

We take the integer kernel K_4 of $I_3 \cdot M_{3,2}$ and keep $I_4 = K_3 \cdot I_3$ which annihilates all the above (M_0, M_2, $M_{2,2}$, M_3) as well as $M_{3,2}$. We give its transpose as

$$I_4^t = \begin{pmatrix} -202 & 422 & -216 & -122 & -232 & -310 \\ -147 & 321 & -389 & -27 & -114 & -387 \\ 99 & -109 & 63 & 67 & 72 & 85 \\ 337 & -729 & 569 & 211 & 394 & 511 \\ 306 & -318 & 216 & 18 & 232 & 206 \\ 605 & -695 & 151 & 369 & 622 & 375 \\ 1016 & -1272 & 682 & 488 & 824 & 824 \\ 186 & -126 & -14 & 66 & 244 & 62 \\ 281 & -525 & 427 & 197 & 306 & 319 \\ 108 & -100 & -14 & 92 & 88 & 36 \\ -73 & 83 & -63 & -41 & -46 & -85 \\ 0 & 0 & -26 & 0 & 0 & 0 \end{pmatrix}.$$

This time the covolume $c_{2,3} \doteq 0.1921035533$ of $I_3 \cdot M_{3,2}$ is a rational multiple of $c_2^2 \cdot c_3$. (From the conjectural framework, we expect this covolume to be a rational number times $c_2^{r_1+r_2} c_3^{r_2}$.)

(X3) *Tetralogarithmic conditions.* Continuing in this way, we determine the tetralogarithmic expressions

$$M_4 \;=\; \big(\alpha(z_\nu)\,\mathcal{L}_4(z_\nu)\big)_\nu \doteq \begin{pmatrix} -0.3814538586 \\ -0.3456381997 \\ 1.759643788 \\ -2.947003507 \\ 2.965401097 \\ 1.431744064 \\ -2.372320878 \\ -2.147616097 \\ 0.9823345025 \\ 0.8798218942 \\ 1.728190998 \\ -1.525815434 \end{pmatrix}.$$

Multiplying I_4 by M_4, we obtain a single column

$$I_4 \cdot M_4 \doteq \begin{pmatrix} -57.01118640 \\ 99.01942902 \\ -67.01314893 \\ -37.00726135 \\ -58.01138266 \\ -71.01393394 \end{pmatrix} \doteq 1.0001962525 \begin{pmatrix} -57.00000000 \\ 99.00000000 \\ -67.00000000 \\ -37.00000000 \\ -58.00000000 \\ -71.00000000 \end{pmatrix}.$$

The covolume $c_4 \doteq 1.0001962525$ is found to be

$$c_4 \doteq \frac{135}{832} \cdot \frac{|d_F|^{7/2}}{\pi^{4(r_1+r_2)}} \cdot \zeta_F(4)\,.$$

The corresponding kernel I_5 (simultaneously satisfying all the conditions (X1-3) and (Y1-2)) can be written as the transpose of

$$I_5^t = \begin{pmatrix} -1 & -8 & 5 & 0 & -3 \\ 7 & 5 & 14 & 0 & -28 \\ 2 & -1 & 0 & 0 & -4 \\ -7 & -2 & 1 & 1 & 12 \\ -3 & 0 & -9 & -16 & -9 \\ 20 & 16 & 6 & -16 & -50 \\ 7 & 2 & -7 & -16 & -39 \\ 10 & 8 & 2 & -16 & -22 \\ -5 & -7 & 8 & 1 & 0 \\ 5 & 2 & 3 & 0 & -13 \\ -1 & 1 & 2 & 0 & 0 \\ 1 & 2 & -1 & 0 & -1 \end{pmatrix}.$$

(D_{10^31}) In the last step we compute the values under the "exotic Bloch-Wigner function" D_{10001} as given above, and arrive at

$$M_5 := \big(D_{10001}(\sigma_\ell\, z_\nu)\big)_{\nu,\ell} \doteq \begin{pmatrix} 0.3798787865 & 4.822987148 \\ 0.3906525699 & 4.688219297 \\ 0.8300465454 & 3.469259751 \\ 1.231915999 & 2.696057038 \\ 1.859803935 & 1.458356504 \\ 0.8801812017 & 3.243252430 \\ 4.330778718 & 1.182477899 \\ 1.346885380 & 2.427729773 \\ 4.871430134 & 1.713967038 \\ 5.141619571 & 1.645178851 \\ 2.153722031 & 1.301307423 \\ 5.247265750 & 1.073768725 \end{pmatrix}.$$

The product of I_5 and M_5 yields the matrix

$$I_5 \cdot M_5 \doteq \begin{pmatrix} 55.64437827 & 108.5338106 \\ 17.97137546 & 44.41609166 \\ 22.97809554 & 115.5346022 \\ -128.5790416 & -128.5790416 \\ -331.9825665 & -424.5390732 \end{pmatrix}$$

which can be decomposed as

$$\doteq \begin{pmatrix} 38.00000000 & 0.00000000 \\ 0.00000000 & 38.00000000 \\ -77.00000000 & 287.0000000 \\ -248.0000000 & 496.0000000 \\ -519.0000000 & 905.0000000 \end{pmatrix} \begin{pmatrix} 1.464325743 & 2.856152912 \\ 0.4729309331 & 1.168844517 \end{pmatrix}.$$

$\zeta_F(5)$: *The special value.* We can compute the covolume $R_{10001}(F)$ of $I_5 \cdot M_5$ as

$$R_{10001}(F) \doteq 13.71063010$$

(which is 38 times 0.3608060553, the latter number being the determinant of the 2×2-matrix on the right). Guided by Borel's theorems, we compare this number $R_{10001}(F)$ with the special value

$$\zeta_F(5) \doteq 1.00041799247384495$$

and we observe (denoting d_F the discriminant of F and r_2 the number of complex embeddings which are in our case -23 and 1, respectively):

Experimental Evidence: For the field F of degree 3 over $\mathbb{Q}$ and of discriminant $d_F = -23$, the five columns $(a_{j,\nu})_\nu$, $1 \le j \le 5$, of I_5^t give rise to non-trivial extension classes $\sum_{\nu=1}^{12} a_{j,\nu}[z_\nu]$ in $\mathrm{Ext}^1_{\mathrm{MTM}/F}(\mathbb{Q}(0), \mathbb{Q}(5))$, and moreover their images under $(D_{10001,\sigma})_\sigma$ generate a lattice of (full) rank 2 and of covolume $R_{10001}(F)$ with

$$\frac{|d_F|^{9/2}}{\pi^{5r_2}} \cdot \zeta_F(5) \;\doteq\; 320 \cdot R_{10001}(F)\,.$$

5.1. Remarks.

(1) The example above is one of the simplest cases that worked, i.e. that gave a non-zero regulator. Usually the number of z_ν which are needed to achieve such a non-zero regulator is considerably larger, and more often than not the program does not find sufficiently many of them.
(2) The procedure of taking the kernel rationally is numerically highly unstable, and since we want to be able to recognize a lattice, it is crucial that we find generators of the kernels which form a $\mathbb{Z}$-basis for the lattice (or which are at least not far away from this property, i.e. the denominator should be bounded). The problem is that, for the matrix entries we encounter, it may take very long to find such a $\mathbb{Z}$-basis already for, say, a lattice given by a matrix of size 300×600. The number of conditions grows very fast with the order of S and it is typically impractical to include more than 3 or 4 primes into S.
(3) One can view the above weight 5 function as a multiple polylogarithm as introduced by Goncharov (see e.g. [**8**]), but specialized to one variable only (then also called "generalized polylogarithm" in the literature), and those multiple polylogarithms in turn have appeared early as "hyperlogarithms", in particular in work of Lappo-Danilevsky [**11**]. In this setting the function above is denoted $Li_{4,1}(1,z)$, and it might be tempting to think of the resulting special value not as a Dedekind zeta value $\zeta_F(5)$ but as a kind of "multiple Dedekind zeta value" $\zeta_F(4,1)$ (for some candidates see Wojtkowiak's original article [**13**]) which then would seem, modulo the product $\zeta_F(2)\zeta_F(3)$, to be a rational multiple of the former, similar to what is known to be true for $F=\mathbb{Q}$ where one has $\zeta(4,1)=5\zeta(5)-\zeta(2)\zeta(3)$. Alas, we were unable to give a sensible evaluation of such a multiple Dedekind zeta value which might have corroborated such a statement.

6. Further results

6.1. Totally real fields. The simplest cases to consider seem to be the ones which are totally real, as the conditions involving dilogarithms and tetralogarithms all are trivially satisfied, since the function $\mathcal{L}_{2n}(z)$ vanishes on the real line. Nevertheless, they turn out to be hard, due to complexity reasons (the integer kernels involved tend to have large coefficients).

6.1.1. *The case* $F=\mathbb{Q}$. In the case of the rational numbers we also find a non-trivial result; using the set $S=\langle 2,3,5\rangle$, we find 98 exceptional S-units, and a similar calculation as above, but where we only need to satisfy conditions (M), followed by (X2) which produces a (numerical) lattice of covolume $\zeta(3)/96$, gives us a lattice generated by the image under D_{10001} of 50 linear combinations in $\mathbb{Z}[F]$, of covolume $\frac{1}{480}\zeta(5)$ (and we get the same lattice if we add the prime 7 to S, yielding 178 exceptional S–units and 87 generators of the lattice).

6.2. Other number fields. We have obtained similar results for number fields of degree ≤ 6, typically we were lucky to find a full regulator lattice for some small discriminants. We list signatures and corresponding discriminants for which we have obtained a (conjectural) lattice of full rank r_1+r_2:

signature	discriminants
$[2,0]$	5, 8, 13
$[0,1]$	-3, -7, -8, -15, -20
$[1,1]$	-31, -44, -59, -76, -83, -104, -108, -116, -139, -152
$[3,0]$	49, 148, 229
$[2,1]$	-275, -283, -331, -400, -448
$[4,0]$	725, 1125
$[0,3]$	-9747.

Moreover, if we divide the covolume of the (conjectural) lattice of D_{10001}–values by $|d_F|^{9/2}\pi^{-5r_2}\zeta_F(5)$, the result in each case is numerically close to a rational number of small height.

For many other number fields, the height (or complexity) of the rational numbers in the inductive steps explodes quickly and gets out of control.

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