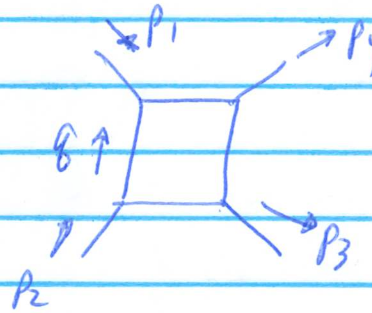
BRIN MRC MARYLAND LECTURES SEP, 2023

CLUSTER ALGEBRAS & POLYLOGARITHMS

1. An invitation to polylogs in physics

p_i = energy-momentum 4-vector of particle i



$p_1 + p_2 = p_3 + p_4$ conservation (and at each vertex)

a "one loop" four particle Feynman diagram

$$\int d^4q \, \frac{1}{q^2 (q+p_1)^2 (q+p_1-p_4)^2 (q-p_2)^2}$$

introduce "dual momentum variables"

$\cancel{X_1 = p_1 - p_2}$

$\cancel{X_2 = p_2 + p_3}$

$\cancel{X_3 = p_3 + p_4}$

$\cancel{X_4 = p_4 + p_1}$

$\cancel{\Rightarrow X_1 + X_2 + X_3 + X_4 = 0}$

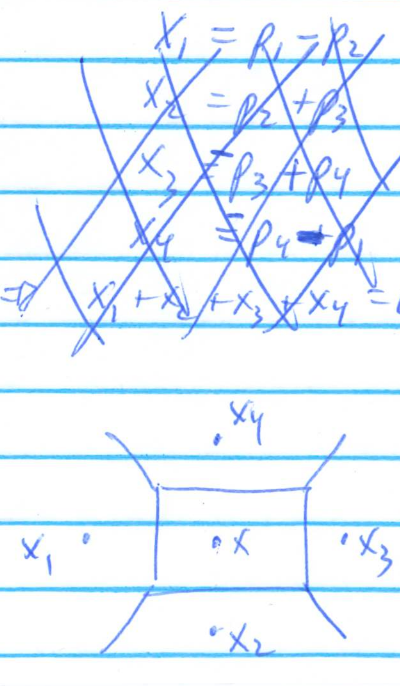
$p_1 = x_1 - x_4$ $\qquad q \to q - x_1$

$p_2 = x_2 - x_1$

$p_3 = x_2 - x_3$

$p_4 = x_3 - x_4$

$\Rightarrow p_1 + p_2 = p_3 + p_4$



$$= \int d^4x \, \frac{1}{(x-x_1)^2 (x-x_2)^2 (x-x_3)^2 (x-x_4)^2}$$

(1) $$\frac{1}{A_1A_2A_3A_4} = 6\int_0^\infty \frac{d\alpha_1\, d\alpha_2\, d\alpha_3}{(\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + A_4)^4}$$

(2) shift $x \to x + \dfrac{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + x_4}{1+\alpha_1+\alpha_2+\alpha_3}$

to eliminate the linear term. The d^4x integral then

$$\int \frac{6}{(Ax^2+\beta)^4}\, d^4x = 12\pi^2 \int_0^\infty \frac{x^3\, dx}{(Ax^2+\beta)^4} = \frac{\pi^2}{A^2\beta^2}$$

(3) Amazingly the last integral is a total derivative and the result can be written as a 2-fold

$$\frac{\pi^2}{(x_1-x_3)^2(x_2-x_4)^2}\int_0^\infty d\alpha_1\, d\alpha_2 \frac{u}{(1+\alpha_1+\alpha_2)(\alpha_1 u + \alpha_2 v + \alpha_1\alpha_2)}$$

$$u = \frac{(x_1-x_3)^2(x_2-x_4)^2}{(x_1-x_4)^2(x_2-x_3)^2} \qquad v = \frac{(x_1-x_2)^2(x_3-x_4)^2}{(x_1-x_4)^2(x_2-x_3)^2}$$

The remaining integral famously evaluates to

$$\frac{\pi^2}{\sqrt{\Delta}}\left[Li_2\left(\frac{z}{\bar z}\right) - Li_2\left(\frac{1-z}{1-\bar z}\right) + Li_2\left(\frac{1-1/z}{1-1/\bar z}\right) - (z \leftrightarrow \bar z)\right]$$

$$z\bar z = u \qquad (1-z)(1-\bar z) = v$$

Note: • except for a simple prefactor $(x_1-x_3)^2(x_2-x_4)^2$
it evaluates to a function of cross-ratios
only, - ie conformally invariant in x-space.

• out of the four original integrals, half
become pi^2 and half give "functional" weight
To find a general algorithm to effect this
This is an outstanding problem

D even $\rightarrow$ $\pi^{D/2}$ (weight $D/2$)

D odd $\rightarrow$ $\pi^{(D\pm1)/2}$. (weight $(D\mp1)/2$)

• the "motivic" amplitude has square roots, $\boxed{\text{How to see?}}$

2. The kinematic space of planar $\mathcal{N}=4$ SYM theory

this particular theory has conformal symmetry in x-space
and its amplitudes are particularly rich mathematically.

Of crucial importance, the ~~theory~~ n-particle amplitude
naturally has a cyclic order of the n-particles.

n ordered points $x_1, x_2, \ldots, x_n$ satisfying $(x_i - x_{i+1})^2 = 0$
(with respect to the metric $-+++$) $\Leftrightarrow$
n ordered points in $\mathbb{P}^3$

Given a point in complexified compactified Minkowski space $M^{1,3}$ x; we associate a line in $\mathbb{P}^3$ by

$$L(x) = \{ Z = (\lambda^1, \lambda^2, \mu^1, \mu^2) : \lambda = x\mu \}$$

$$x = \begin{pmatrix} x_0 + x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_0 - x_3 \end{pmatrix}$$

Two lines $L(x_1)$, $L(x_2)$ intersect in $\mathbb{P}^3 \iff$ $x_1 - x_2$ is null.

$\Leftarrow$ $\exists\ \lambda, \mu$ s.t. $\lambda = x_1 \mu \quad \lambda = x_2 \mu \Rightarrow 0 = (x_1 - x_2)\mu$
$\Rightarrow \det(x_1 - x_2) = 0$

A configuration of n particles is specified by

$$\begin{pmatrix} | & & | \\ z_1 & \cdots & z_n \\ | & & | \end{pmatrix}$$

an $n \times 4$ matrix containg the homogeneous coords $z_i \in \mathbb{P}^3$.

A special symmetry of $N=4$ SYM theory is dual confomal symmetry which is PGL_4 acting on the left so we really have a quotient of the Grassmannian

$$G(4,n) / (\mathbb{C}^*)^{n-1}$$

Amplitudes in $N=4$ SYM are functions on this space

3. DDS. → GSVV

Thanks to this symmetry the simplest nontrivial amplitudes have $n=6$ (cant make cross ratios $n<6$)

CROSS RATIOS

one loop (weight 2) is a little too trivial you only get $\log^2$ terms.

Really have to go to 2 loops

DDS p100

DDS p 15

DDS p34

$$G(a; z) = \ln(1 - z/a) \qquad a \neq 0$$

$$G(a, \vec{w}; z) = \int_0^z \frac{dt}{t-a} G(\vec{w}, t)$$

$$G(\underbrace{0, \ldots, 0}_{n}; z) = \frac{1}{n!} \log^n z$$

there are 3 cross ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2} \qquad u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} \qquad u_3 = \frac{x_{26}^2 x_{35}^2}{x_{23}^2 x_{36}^2}$$

If this is the simplest nontrivial amplitude, there is no hope!

4. The Hopf Algebra of Multiple Polylogarithms.

Iterated integrals

$$I(a_0;\ a_1, \dots, a_n;\ a_{n+1})$$

$$= \int_{a_0}^{a_{n+1}} \frac{dt_1}{t_1 - a_1} \int_{a_0}^{t_1} \frac{dt_2}{t_2 - a_2} \cdots \int \frac{dt_n}{t_n - a_n}$$

These iterated integrals are elements of a Hopf algebra graded by weight (n).

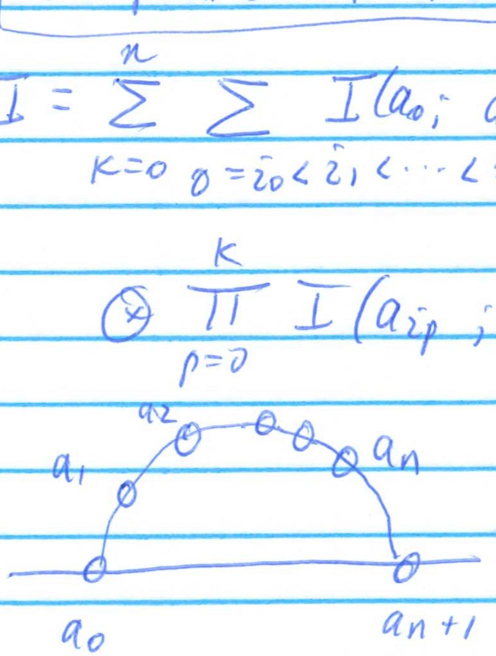
$$\Delta : A \to A \otimes A \text{ such that}$$

$$\Delta(I_1 \cdot I_2) = \Delta(I_1) \cdot \Delta(I_2)$$

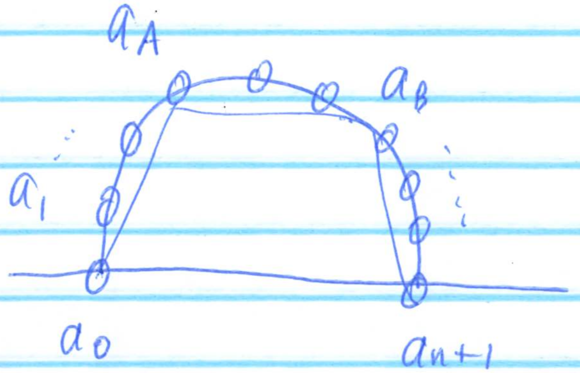
- $\boxed{\text{Compatible with multiplication!}}$ $\boxed{\cdot \text{ coassociative}}$

$$\Delta I = \sum_{k=0}^{n} \sum_{0 = i_0 < i_1 < \cdots < i_{k+1} = n+1} I(a_0;\ a_{i_1}, \dots, a_{i_k};\ a_{n+1})$$

$$\otimes \prod_{p=0}^{k} I(a_{i_p};\ a_{i_p+1}, \dots a_{i_{p+1}-1};\ a_{i_{p+1}})$$



[Field of rational functions on $\mathrm{conf}_n(\mathbb{P}^3)$]



Ex a term with $k=2$

$$\underbrace{I(a_0; a_A, a_B; a_{n+1})}_{\text{weight } 2} \otimes \overbrace{I(a_0; a_1, \ldots, a_{A-1}; a_A)}^{a_A - 1} \cdot \overbrace{I(a_A; a_{A+1}, \ldots, a_{B-1}; a_B)}^{a_B - a_A - 1} \cdot \underbrace{I(a_B; a_{B+1}, \ldots, a_n; a_{n+1})}_{n - a_B}$$

total $n-2$

$$\Delta \mathrm{Li}_n(z) = 1 \otimes \mathrm{Li}_n(z) + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!} + \mathrm{Li}_n(z) \otimes 1$$

$\mathcal{L} = \mathcal{A} / (\mathcal{A}_{>0} \cdot \mathcal{A}_{>0})$ = algebra of iterated integrals modulo products.

The restriction of Δ to $\mathcal{L}$ is called the cobracket δ. It satisfies $\delta^2 = 0$ giving $\mathcal{L}$ the structure of a Lie coalgebra.

· The $1,1,1,\ldots,1$ component of
The maximum iteration of the coproduct is called
the $\boxed{\text{symbol}}$

$$A \xrightarrow{\Delta} A\otimes A \xrightarrow{\Delta\otimes 1} A\otimes A\otimes A \rightarrow \ldots$$

e.g. $\mathcal{S}(Li_2(z)) = -(1-z)\otimes z$
$\mathcal{S}(\log a \log b) = a\otimes b + b\otimes a$

We can compute the symbol of DDS.
Use $Gr(4,6)$ notation

$$u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle} \quad \text{etc} \quad u_2, u_3 \text{ cyclic}$$

- The u_{ijk}'s simplify; they become rational in Pluckers
- the v's drop out.

- The amount of cancellation is huge
- The symbol alphabet = $Gr(4,6)$ Plucker coordinates.

Next observation is that the 2,2 component of the cobracket vanishes $\rightarrow$ the function is expressible in terms of Li_4 only.

$\Rightarrow$ GSVV formula 29 June 2010

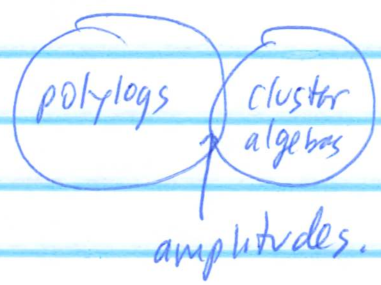
5. Cluster Algebras

By 27 May 2011 (< 1 year) all two loop n-particle MHV amplitudes were known
To explain them we need another tool

<u>Teaser</u>

$$\begin{cases} x_1,\ x_2, & x_{i-1}x_{i+1} = 1 + x_i \quad \text{exchange relation} \\ x_3 = \dfrac{1+x_2}{x_1} & \\ x_4 = \dfrac{1+x_1+x_2}{x_1 x_2} & \\ x_5 = \dfrac{1+x_1}{x_2} & x_6 = x_1 \end{cases}$$

A_2 cluster algebra $\quad \sum_{i=1}^{5} D_2(-x_i) = 0 \quad$ Abel

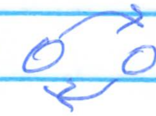


Cluster algebra = commutative algebra with a collection of preferred generators called cluster coords assembled into sets of equal cardinality called clusters

labeled quivers

no loops 

$$x_1 \longrightarrow x_2 \longrightarrow x_3$$

no 2-cycles 

there is a rule called mutation for generating new cluster variables

- pick a node j
- for each $i \to j \to k$ add $i \to k$
- reverse all lines in/out j
- delete all 2 cycles

how do variables change? let B_{ij} = # arrows $i \to j$

if you mutate on j

$$x_i' = \begin{cases} 1/x_i & \text{if } i = j \\ x_i\left(1 + x_j^{\operatorname{sgn} B_{ij}}\right)^{B_{ij}} & \text{if } i \neq j \end{cases}$$

THIS IS AN INVOLUTION

if $\{\log x_i, \log x_j\} = B_{ij}$ then $\{\log x_i', \log x_j'\} = B_{ij}'$

Fock & Goncharov

Who are the 770 cluster variables for $Gr(3,7)$

$$\frac{\langle 125\rangle\langle 7\times 2, 3\times 4, 5\times 6\rangle}{\langle 257\rangle\langle 1\times 2, 3\times 4, 5\times 6\rangle} = 1 + \frac{\langle 127\rangle\langle 256\rangle\langle 345\rangle}{\langle 257\rangle\langle 1\times 2, 3\times 4, 5\times 6\rangle}$$

$$\frac{\langle 1\times 2\rangle\langle 3\times 4\rangle\langle 5\times 7\rangle\langle 7\times 2, 3\times 4, 5\times 6\rangle}{\langle 257\rangle\langle 347\rangle\langle 1\times 2, 3\times 4, 5\times 6\rangle} = 1 + \frac{\langle 127\rangle\langle 234\rangle\langle 345\rangle\langle 567\rangle}{\langle 257\rangle\langle 347\rangle\langle 1\times 2, 3\times 4, 5\times 6\rangle}$$

are some of the most complicated.

Brief Review

Goncharov, MS, Vergu, Volovich (2010):

The 2-loop 6-particle MHV amplitude

(1) has a symbol alphabet = $Gr(2,6)$ Plucker coordinates

(2) can be expressed entirely in terms of Li_m $m=1,2,3,4$

By 2011, the 2-loop MHV amplitudes had been computed for all n. [Caron-Huot]

(2) is an accident that is not true of any more complicated amplitudes, but (1) is just a tip of an iceberg; there is much deeper cluster structure.

5. Cluster Algebras

It will be important for me to clarify that there are 2 types of cluster variables — $\mathcal{A}$ & $\mathcal{X}$.

In the Grassmannian case,

$\mathcal{A}$ = (polynomials in) Plückers

$\mathcal{X}$ = CERTAIN homogeneous ratios of $\mathcal{A}$s

seeds for $\mathcal{A}$ seeds for $\mathcal{X}$

$A_2 = G(2,5)$ example

$$\begin{tikzcd}
\boxed{\langle 12\rangle} \arrow[dr] & & & \\
& \langle 13\rangle \arrow[r] \arrow[d] & \langle 14\rangle \arrow[r] \arrow[d] & \boxed{\langle 15\rangle} \\
& \boxed{\langle 23\rangle} & \boxed{\langle 34\rangle} \arrow[ul] & \boxed{\langle 45\rangle} \arrow[ul]
\end{tikzcd}$$

$$\frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle} \longrightarrow \frac{\langle 13\rangle\langle 45\rangle}{\langle 34\rangle\langle 15\rangle}$$

The rules for mutating in the $\mathcal{A}$-picture were explained by Lauren.

In the $\mathcal{X}$ picture, the rules are

- the same, as far as manipulating the quiver (we just ignore frozen nodes)

- on cluster variables, when you mutate on X_K,

$$X_K \to X_K^{-1}$$

$$X_i \to X_i \left(1 + X_K^{\text{sign } B_{iK}}\right)^{B_{iK}} \quad \text{if } i \neq K$$

where $B_{iK} =$ (signed) # arrows from $i \to K$

In the above example, if we mutate on the

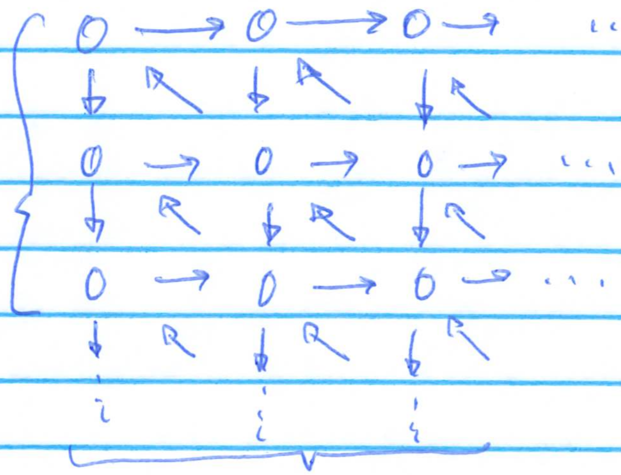
left: $\dfrac{\langle 23\rangle\langle 14\rangle}{\langle 12\rangle\langle 34\rangle} \longleftarrow \dfrac{\langle 12\rangle\langle 45\rangle}{\langle 24\rangle\langle 15\rangle}$

right: $\dfrac{\langle 12\rangle\langle 35\rangle}{\langle 23\rangle\langle 15\rangle} \longrightarrow \dfrac{\langle 34\rangle\langle 15\rangle}{\langle 13\rangle\langle 45\rangle}$

require a short calculation using Plücker relations to simplify

The $G(k,n)$ cluster algebra has a seed of the form (Scott 2006)

$k-1$ rows



the variable @ each node is a cross-ratio of Plückers

$n-k-1$ columns

cluster algebra	alt. name	#clusters	#$\mathcal{A}$-vars (excluding frozen)	#$\mathcal{X}$-vars (counting $x, \frac{1}{x}$ once)
$Gr(2,4)$	A_1	2	2	1
$Gr(2,5)$	A_2	5	5	5
$Gr(2,6)$	A_3	14	9	15
$Gr(2,n)$	A_{n-3}	Catalan	$\left(\frac{n(n-3)}{2}\right)$	$\binom{n}{4}$
$Gr(3,6)$	D_4	50	16	52
$Gr(3,7)$	E_6	833	42	385
$Gr(3,8)$	E_8	25080	128	3120

Sherman-Bennett 2018

all other $G(k,n)$'s are infinite

Every $\mathcal{X}$-var is a Laurent monomial in $\mathcal{A}$s.

Comment

Moreover, every homogeneous Laurent monomial in $\mathcal{A}$s can be written (nonuniquely!) as a Laurent monomial in $\mathcal{X}$s.

6. TWO-LOOP MHV DATA

For all n, the symbol of the 2-loop MHV n-particle amplitude can be expressed in terms of a (subset) of cluster $\mathcal{A}$-variables of $Gr(4,n)$

In fact only certain Plückers and certain quadratic polynomials in Pluckers appear, no higher polynomials.
(For $n=6,7$ <u>all</u> cluster $\mathcal{A}$-variables appear.)

Based on this data we (Golden, Paulos, MS, Volovich 2014) defined a notion of "cluster polylogarithm" that is both <u>weaker</u> and <u>stronger</u> than Rudenko's definitions

<u>Weaker</u> because we do <u>not</u> impose the condition that every term in the symbol must involve four compatible $\mathcal{A}$-variables $a_1 \otimes a_2 \otimes a_3 \otimes a_4$
— we allow them to live in arbitrary clusters.

Let $\mathcal{H}_A$ = Hopf algebra of multiple polylogs whose symbols are homogeneous and have alphabet = A variables
(some finite set of

But stronger because
let $\mathcal{H}_X$ = Hopf algebra of multiple polylogs "whose arguments are $-X$ variables"

(Recall $\mathcal{L}_{A,X} = \mathcal{H}_{A,X}$ ~~modulo products~~)

Weight 1. $\mathcal{H}_{A,1} = \mathcal{H}_{X,1}$ by comment above

$$\{\mathbb{Q} \text{ span of } \log A\}/\text{homogeneous} = \{\mathbb{Q} \text{ span of } \log X\}$$

Weight 2. $\mathcal{L}_{A,2} = \mathcal{L}_{X,2}$

$\mathbb{Q}$ span of $Li_2(-X)$ gives everything (can check).

Weight 3. $\mathcal{L}_{A_{,3}}$ is "twice as big" as $\mathcal{L}_{X_{,3}}$!

For any x-coordinate x,

$Li_3(-x)$ and $Li_3(1+x)$ are both in $\mathcal{L}_A$ but only the first is in $\mathcal{L}_X$.

$\Rightarrow$ D. Rudenko had the same conclusion in his construction, but from a different starting point!

Weight 4. We need a definition

Define $\mathcal{X}$-cluster function of weight 4 as

$$\delta_{2,2} f = \text{linear comb of } Li_2(-x_i) \wedge Li_2(-x_j)$$
where x_i, x_j are compatible

$$\delta_{3,1} f = \text{'' '' '' } Li_3(-x_i) \wedge Li_1(-x_j)$$
where x_i, x_j are compatible

$\mathcal{L}_{\mathcal{X},4}$ is much smaller than $\mathcal{L}_{A,4}$!

Amazingly seems to equal (from his tables) Rudenko's definition!

All two loop MHV amplitudes are cluster $\mathcal{X}$-functions in this sense.

Much more is true.

7. GSV CLUSTER POISSON VARIETY

Cluster $\mathcal{X}$ coordinates have a natural Poisson bracket

$$\{\log x_i, \log x_j\} = B_{ij}$$ compatible with mutation

$$\{\log x_i', \log x_j'\} = B_{ij}'$$

For all two loop amplitudes

$\delta_{2,2} A$ = linear comb of $Li_2(-x_i) \wedge Li_2(-x_j)$
with $\{\log x_i, \log x_j\} = 0$

$\delta_{3,1}$ = linear comb of $Li_3(-x_i) \wedge Li_1(-x_j)$ }
with $\{\log x_i, \log x_j\} = \pm 1.$