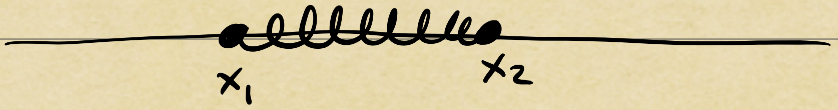
3. Two beads of unit mass move along a straight horizontal wire, without friction, with positions x_1 and x_2 (we will assume $x_2 > x_1$). We join them with a spring of natural length a and constant $\kappa = 1$.

(a) Introduce the generalised coordinates $q_1 = x$ and $q_2 = x_2 - a$. Show that the Lagrangian describing the system in these coordinates is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}(q_2 - q_1)^2 .$$



$$\left.\begin{array}{l} q_1 = x_1 \\ q_2 = x_2 - a \end{array}\right\} \begin{array}{l} \dot{q}_1 = \dot{x}_1 \\ \dot{q}_2 = \dot{x}_2 \end{array}$$

In the x_1, x_2 coordinate system:

$$T = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2$$

$$V = \frac{1}{2}\kappa(\ell)^2 = \frac{1}{2}\kappa(x_2 - x_1 - a)^2$$

ℓ: extension relative to the natural length

$\kappa = 1$

$$= \frac{1}{2}(q_2 - q_1)^2$$

$q_2 = x_2 - a$
$q_1 = x_1$

$$\Rightarrow L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - \frac{1}{2}(q_2 - q_1)^2$$

(b) Write the Euler-Lagrange equations of motion for q_1 and q_2 coming from this Lagrangian.

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \ddot{q}_1 + q_1 - q_2$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \ddot{q}_2 - q_1 + q_2$$

(c) Find the normal modes of the system, including possible zero modes, and write the general solution $q_1(t)$ and $q_2(t)$ for the motion of the system in terms of these normal modes.

We write the equation of motion in matrix form:

$$\frac{d^2}{dt^2}\vec{q} + A\vec{q} = 0 \quad \text{with} \quad \vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$\text{and} \quad A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Characteristic polynomial:

$$p_A(\lambda) = \det(A - \lambda) = (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$$

Eigenvalues are $\lambda = 2$, $\lambda = 0$

with eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ +1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = b$$

The general solution

$$\begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{\lambda = 2}\left[\alpha \cos(\sqrt{2}\,t) + \beta \sin(\sqrt{2}\,t)\right]$$

$$+ \begin{pmatrix} 1 \\ 1 \end{pmatrix}\left[Ct + D\right]$$

(d) Assume that the system starts at rest, with $q_2 = -q_1 = d > 0$. Find the subsequent motion of the system, that is, give explicit expressions for $q_1(t)$ and $q_2(t)$ compatible with the given initial conditions.

$$\begin{pmatrix} q_1(0) \\ q_2(0) \end{pmatrix} = \begin{pmatrix} -d \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} [\alpha] + \begin{pmatrix} 1 \\ 1 \end{pmatrix} D$$

$$= \begin{pmatrix} \alpha + D \\ -\alpha + D \end{pmatrix}$$

$$\Rightarrow \quad \alpha = -d$$
$$D = 0$$

"Starting at rest" : $\begin{pmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \beta\sqrt{2} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} C = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \beta\sqrt{2} + C \\ -\beta\sqrt{2} + C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \beta = C = 0$$

So the final solution is

$$\vec{q}(t) = \begin{pmatrix} -d \\ d \end{pmatrix} \cos(\sqrt{2}\, t)$$

8. Consider an infinitesimal transformation of the generalised coordinates of the form

$$q_i \to q_i' = q_i + \epsilon a_i(q_1, \ldots, q_n) \qquad ; \qquad \dot{q}_i \to \dot{q}_i' = \dot{q}_i + \epsilon \dot{a}_i(q_1, \ldots, q_n)$$

where we have dropped possible terms of quadratic and higher order in the infinitesimal parameter ϵ.

(a) Show, using the Euler-Lagrange equations, that under such a transformation the change in the Lagrangian

$$\delta L = L' - L = L(q_1', \ldots, q_n', \dot{q}_1', \ldots, \dot{q}_n') - L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

is given by

$$\delta L = \epsilon \frac{dQ}{dt} + O(\epsilon^2)$$

for some Q that you should find explicitly. In the special case $\delta L = O(\epsilon^2)$ the transformation is a symmetry, and the Q that you found is a conserved charge.

Doing a Taylor expansion:

$$L(q_1', \ldots, q_n', \dot{q}_1', \ldots, \dot{q}_n') = L(q_1 + \varepsilon a_1, q_2 + \varepsilon a_2, \ldots, \dot{q}_1 + \varepsilon \dot{a}_1, \ldots)$$

$$= L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n) + \varepsilon \left[\sum_{i=1}^{n} \frac{\partial L}{\partial q_i} a_i + \frac{\partial L}{\partial \dot{q}_i} \dot{a}_i\right] + O(\varepsilon^2) \underset{\uparrow}{=} L + \varepsilon \sum_{i=1}^{n} \left(a_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \dot{a}_i \frac{\partial L}{\partial \dot{q}_i}\right) + O(\varepsilon^2)$$

E-L: $\dfrac{\partial L}{\partial q_i} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right)$

$$= L + \varepsilon \sum_{i=1}^{n} \frac{d}{dt}\left(a_i \frac{\partial L}{\partial \dot{q}_i}\right) + O(\varepsilon^2)$$

$$= L + \varepsilon \frac{d}{dt}\left(\sum_{i=1}^{n} a_i \frac{\partial L}{\partial \dot{q}_i}\right) + O(\varepsilon^2)$$

Q

(b) Consider a Lagrangian of the form

$$L_{ab} = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - aq_1^2 - bq_2^2 .$$

Find the relation that needs to be satisfied between a and b so that the infinitesimal rotation

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

is a symmetry of L_{ab}.

$$\left.\begin{array}{l} q_1 \to q_1 + \varepsilon q_2 \\ q_2 \to q_2 - \varepsilon q_1 \end{array}\right\} \begin{array}{l} q_1^2 \to q_1^2 + 2\varepsilon q_1 q_2 + O(\varepsilon^2) \\ q_2^2 \to q_2^2 - 2\varepsilon q_1 q_2 + O(\varepsilon^2) \end{array}$$

and similarly for the dotted variables (to first order in ε):

$$\dot{q}_1^2 \to \dot{q}_1^2 + 2\varepsilon \dot{q}_1 \dot{q}_2 + O(\varepsilon^2)$$

$$\dot{q}_2^2 \to \dot{q}_2^2 - 2\varepsilon \dot{q}_1 \dot{q}_2 + O(\varepsilon^2)$$

$$L_{ab} \to L_{ab} + 2\varepsilon(b-a) q_1 q_2 + O(\varepsilon^2)$$

<u>Symmetry</u>: a transformation that acts on the Lagrangian:

$$L \to L + \varepsilon \frac{dF}{dt} + O(\varepsilon^2)$$

for some F.

This is a symmetry iff $a = b$.

(c) Denote by L_a the Lagrangian with a arbitrary, but b chosen so that the rotation above is a symmetry. Compute the conserved charge Q associated to the rotation of L_a.

We have $a_1 = q_2$, $a_2 = -q_1$, so

$$Q = \sum_{i=1}^{n} a_i \frac{\partial L}{\partial \dot{q}_i} = q_2 \dot{q}_1 - q_1 \dot{q}_2$$

(d) Compute the generalised momenta p_i for the Lagrangian L_{ab}, and check that the variation of q_i under the transformation generated by Q, obtained by computing the relevant Poisson bracket, agrees with the transformation you started with.

By definition

$$p_i := \frac{\partial L}{\partial \dot{q}_i} \Rightarrow p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = \dot{q}_2$$

In Hamiltonian language:

$$Q = q_2 p_1 - q_1 p_2$$

$$q_1 \to q_1 + \varepsilon \{q_1, Q\} + O(\varepsilon^2)$$

$$\{q_1, Q\} = \{q_1, q_2 p_1 - q_1 p_2\}$$

Linearity:

$$= \{q_1, q_2 p_1\} - \{q_1, q_1 p_2\}$$

$$= \{q_1, q_2\} p_1 + \{q_1, p_1\} q_2 - \{q_1, q_1\} p_2 - \{q_1, p_2\} q_1$$

Elementary Poisson brackets: $\{q_i, p_j\} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$\{q_i, q_j\} = \{p_i, p_j\} = 0 \quad \forall i, j.$

$\Rightarrow \{q_1, Q\} = q_2$

Similarly: $\{q_2, Q\} = -q_1$

(e) Write down the expression for the Hamiltonian H_{ab} in terms of q_i and p_i. Then show, by computing the Poisson bracket $\dot{Q} = \{Q, H_{ab}\}$, that Q is conserved if and only if the relation you found between a and b above holds.

The Hamiltonian is

$$H_{ab} = \left(\sum_{i=1}^{2} p_i \dot{q}_i\right) - L = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + a q_1^2 + b q_2^2$$

From the properties of the Poisson bracket we find:

$$\{Q, H_{ab}\} = \{q_2 p_1 - q_1 p_2, H_{ab}\}$$

$$= \{q_2, H_{ab}\} p_1 + q_2 \{p_1, H_{ab}\} - \{q_1, H_{ab}\} p_2 - q_1 \{p_2, H_{ab}\}$$

We have

$$\{q_1, H_{ab}\} = \{q_1, \tfrac{1}{2}p_1^2 + \dots\} = p_1 \{q_1, p_1\} = p_1$$

where we have omitted terms whose Poisson bracket with q_1 vanishes. Similarly:

$$\{q_2, H_{ab}\} = p_2$$

$$\{p_1, H_{ab}\} = \{p_1, a q_1^2 + \dots\} = -2a q_1$$

$$\{p_2, H_{ab}\} = -2b q_2$$

Collecting terms:

$$\dot{Q} = \{Q, H_{ab}\} = p_2 p_1 + q_2(-2a q_1) - p_1 p_2 - q_1(-2b q_2)$$

$$= 2 q_1 q_2 (b - a)$$