Now, for the systems that we will study during this term, it will be the case that S can be expressed in a particularly nice way as the time integral of a *Lagrangian*. That is, we will have

$$S[x] = \int_{t_0}^{t_1} dt\, L(x(t), \dot{x}(t)) \tag{2.1.2}$$

for some function $L(a, b)$ of two real variables, where $\dot{x}(t) := \frac{dx}{dt}$.

Whenever a Lagrangian exists, the variational principle together with the fundamental lemma of the calculus of variations leads to a set of differential equations that determine $x(t)$. The argument is as follows. If we Taylor expand the perturbed Lagrangian to first order in $\delta x(t)$ we get[4]

$$L(x(t) + \delta x(t), \dot{x}(t) + \delta\dot{x}(t)) = L(x(t), \dot{x}(t)) + \frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\delta\dot{x}(t) + \ldots$$

Putting this expansion of the Lagrangian into the variation of the action we have

$$\begin{aligned} \delta S &= \int_{t_0}^{t_1} dt\, L(x(t) + \delta x(t), \dot{x}(t) + \delta\dot{x}(t)) - L(x(t), \dot{x}(t)) \\ &= \int_{t_0}^{t_1} dt\, \left(\frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\delta\dot{x}(t)\right) \end{aligned} \tag{2.1.3}$$

where we have omitted terms of second order or higher in δx.[5] For notational simplicity I will often write $\partial L/\partial x$ instead of the more precise but much more cumbersome

$$\left.\frac{\partial L(r,s)}{\partial r}\right|_{(r,s)=(x(t),\dot{x}(t))}$$

where (r, s) are names for the two arguments of the Lagrangian L (which are conventionally, but somewhat confusingly, also named x and $\dot{x}$, a convention that I will follow most of the time... but here I want to be as clear as possible about what I mean). Similarly

$$\frac{\partial L}{\partial \dot{x}} := \left.\frac{\partial L(r,s)}{\partial s}\right|_{(r,s)=(x(t),\dot{x}(t))}.$$

[4] To bring the main point to light here: note that from the point of view of the Lagrangian $x(t)$, $\delta x(t)$, $\dot{x}(t)$ and $\delta\dot{x}(t)$ are simply numbers, not functions. Let me call them a, $\epsilon\alpha$, b and $\epsilon\beta$, respectively, to emphasize this point, where $a, \alpha, b, \beta \in \mathbb{R}$, and $\epsilon \in \mathbb{R}$ is as in definition 2.1.1. Then all we are doing here is taking the first order in the Taylor expansion of the Lagrangian in ϵ:

$$L(a + \epsilon\alpha, b + \epsilon\beta) = L(a, b) + \epsilon\alpha\left.\frac{\partial L(r,s)}{\partial r}\right|_{(r,s)=(a,b)} + \epsilon\beta\left.\frac{\partial L(r,s)}{\partial s}\right|_{(r,s)=(a,b)} + \ldots$$

[5] Recall from definition 2.1.1 that any time we talk about expanding on δx we are really expanding on a small parameter ϵ inside $\delta x(t) = \epsilon z(t)$ (where $z(t)$ is as in definition definition 2.1.1). The variation $\delta\dot{x}(t) = \epsilon\dot{z}(t)$ clearly has the same dependence on ϵ, since ϵ is just a constant that does not depend on time. We therefore have that "$\delta\dot{x}(t)$ is first order in $\delta x(t)$", at least for the purposes of counting degrees when expanding.

We proceed by noting that $\delta\dot{x}(t) = \frac{d}{dt}(\delta x(t))$, so we can write the above as

$$\delta S = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x}\delta x(t) + \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}\delta x(t) \right) .$$

Integration by parts of the second term[6] now allows us to rewrite this as

$$\delta S = \int_{t_0}^{t_1} dt \left[\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x(t) + \frac{d}{dt}\left(\delta x(t)\frac{\partial L}{\partial \dot{x}}\right)\right] .$$

The last term is a total derivative, so we can integrate it trivially to give:

$$\delta S = \left[\delta x(t)\frac{\partial L}{\partial \dot{x}}\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x(t) .$$

Now we have that $\delta x(t_0) = \delta x(t_1) = 0$, as the paths that we consider all start and end on the same positions. This implies that the first term vanishes, so

$$\delta S = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x(t) .$$

Recall that the action principle demands that this variation cancels (to first order in $\delta x(t)$, i.e. ignoring possible terms that we have not written) for arbitrary $\delta x(t)$. By the fundamental lemma of the calculus of variations, the only way that this could possibly be true is if the function multiplying $\delta x(t)$ in the integral vanishes:

$$\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 .} \tag{2.1.4}$$

This is known as the *Euler-Lagrange equation*, in the case of one-dimensional problems.

Note 2.1.5

There is a somewhat subtle point in the Lagrangian formulation that I want to make explicit. Note that the Lagrangian L is an ordinary function of two parameters, and it knows nothing about paths. (In general it is a function of $2N$ parameters, with N the number of "generalised coordinates" that we need to describe the system, see

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[6]Recall that

$$\frac{d(f(t)g(t))}{dt} = \frac{df(t)}{dt}g(t) + f(t)\frac{dg(t)}{dt}$$

or equivalently

$$f(t)\frac{dg(t)}{dt} = \frac{d(f(t)g(t))}{dt} - \frac{df(t)}{dt}g(t) .$$

In the text we have

$$f(t) = \frac{\partial L}{\partial \dot{x}} \qquad ; \qquad g(t) = \delta x(t) .$$

below.) Let me emphasize this by writing $L(r, s)$. When using the Lagrangian function to construct the action we evaluated the Lagrangian function at $(r, s) = (x(t), \dot{x}(t))$ at each instant in time, but it is important to keep in mind that the Lagrangian itself treats r and s as independent variables: they are simply the two arguments to the function.

In general, if we want to study how this function changes under small displacements of r and s we would use the chain rule:

$$L(r + \delta r, s + \delta s) = L(r, s) + \frac{\partial L}{\partial r}\delta r + \frac{\partial L}{\partial s}\delta s + \ldots$$

where the dots denote terms of higher order in δr and δs. This is what we did above in (2.1.3), again with $(r, s) = (x(t), \dot{x}(t))$.

What this all means is that the partial derivatives appearing in the Euler-Lagrange equations treat the first and second arguments of the Lagrangian function independently, leading to the somewhat funny-looking rules:

$$\frac{\partial x}{\partial \dot{x}} = \frac{\partial \dot{x}}{\partial x} = 0\,. \tag{2.1.5}$$

This would probably be a little clearer if we used a different notation for $\dot{x}$ (such as v) when writing Lagrangians, to emphasize that in the Lagrangian formalism $\dot{x}$ should be treated as a variable which is entirely independent of x itself. But I will stick to the standard (if somewhat puzzling at first) notation, with the understanding that in the Lagrangian formalism one should impose (2.1.5).

This also makes clear that (2.1.5) is not something you should generically expect to hold outside the Lagrangian formalism. And indeed, when we study the Hamiltonian framework below this rule will be replaced by a different one.