§2.2 Configuration space and generalized coordinates

We now want to extend the Lagrangian formalism to deal with more general situations, beyond the rather special case of a particle moving in one dimension. We start with

Definition 2.2.1. The set of all possible (in principle) instantaneous configurations for a given physical system is known as *configuration space*. We will denote it by $\mathcal{C}$.

It is important to note that this includes positions, but it does not include velocities. One informal way of thinking about configuration space is as the space of all distinct photographs one can take of the system, at least in principle.

Additionally, in constructing configuration space we make no statement about dynamics: we need to construct configuration space **before** we construct a Lagrangian, which tells us about the dynamics in this configuration space.

Example 2.2.2. *A particle moving in $\mathbb{R}^d$ (that is, d-dimensional euclidean space) has configuration space $\mathbb{R}^d$. We discussed the $d = 1$ example above, where we had a particle moving in the one dimensional line $\mathbb{R}$, which we parametrized by the coordinate x.*

Example 2.2.3. *N particles moving freely in the $\mathbb{R}^d$ have configuration space $(\mathbb{R}^d)^N = \mathbb{R}^{dN}$. (Assuming that we can always distinguish the particles. I leave it as an amusing exercise to think what is the configuration space if you cannot distinguish the particles.)*

Example 2.2.4. *N electrically charged particles moving in $\mathbb{R}^d$: since particles are electrically charged they repel or attract. But this is a dynamical question, and configuration space is insensitive to such matters, it is still $\mathbb{R}^{dN}$. One way to see this is that you can always place a set of N particles at any desired positions in $\mathbb{R}^d$ (barring some singular points where particles overlap, so the system has infinite energy and is unphysical, but we can ignore such subtleties here). After being released, the particles will subsequently move in a manner described by the Euler-Lagrange equations, but any initial choice is permitted.*

Example 2.2.5. *Two particles joined by a rigid rod of length ℓ in d-dimensions: without the rod the configuration space is $\mathbb{R}^{2d}$, but the rod introduces the constraint that the particles are at fixed distance ℓ from each other. This can be written as*

$$||\vec{x}_1 - \vec{x}_2||^2 = \ell^2$$

where $\vec{x}_1$ and $\vec{x}_2$ are the positions of the two particles in $\mathbb{R}^d$. The configuration space is $2d - 1$ dimensional, given by the surface defined by this equation inside $\mathbb{R}^{2d}$.

Example 2.2.6. *Finally, consider a rigid body in $\mathbb{R}^3$, a desk for instance. We can view it as formed by 10^{27} atoms, joined by atomic forces. But for the purposes of classical mechanics we certainly do not care about the motion of the individual atoms (we are doing classical mechanics, not quantum mechanics, so we would get the answer wrong anyway, even if we could compute it!). Rather, for classical dynamics we can think about the classical configurations that the desk can take. And this is a six-dimensional space, given (for instance) by the position of the centre of mass of the desk, and three rotational angles.*

Definition 2.2.7. Given a configuration space $\mathcal{C}$ for a physical system $\mathcal{S}$, we say that $\mathcal{S}$ has $\dim(\mathcal{C})$ *degrees of freedom.*

Although it is illuminating to think of configuration space abstractly, in practice we will often want to put coordinates on it, so that we can write and analyse concrete equations. I emphasize that this is a matter of convenience: any choice of coordinate system is equally valid, and the Lagrangian formalism holds regardless of the choice.

Definition 2.2.8. Given a configuration space $\mathcal{C}$, any set of coordinates in this space is known as a set of *generalized coordinates.* Conventionally, when we want to indicate that some equation holds for arbitrary choices of generalized coordinates, we will use "q_i" for the coordinate names, with $i \in \{1, \ldots, \dim(\mathcal{C})\}$, and "$\mathbf{q}$" (without indices) for the coordinate vector with components q_i.

Example 2.2.9. *Consider the case of a particle moving on $\mathbb{R}^2$. The configuration space is $\mathbb{R}^2$. There are two natural sets of coordinates in this space (although I emphasize again that **any** choice is valid): we could choose the ordinary Cartesian coordinates (x, y), or it might be more convenient to choose* polar coordinates *r, θ satisfying*

$$x = r\cos(\theta)\,,$$
$$y = r\sin(\theta)\,.$$

Example 2.2.10. *Consider instead the case of a bead attached to a circular wire of unit radius on $\mathbb{R}^2$, defined by the equation $x^2 + y^2 = 1$. The configuration space is the circle, S^1. A possible coordinate in this space is the angular variable θ appearing in the description of the circle in polar coordinates.*

Note 2.2.11

We will only be dealing with *unconstrained* generalized coordinates when describing configuration space. That is, we want a set of exactly $\dim(\mathcal{C})$ coordinates (and no more) that describes, at least locally, the geometry of the configuration space $\mathcal{C}$. So in example 2.2.10 we can take θ as our generalized coordinate, but we do not want to consider (x, y) as generalized coordinates, as they are subject to the constraint $x^2 + y^2 = 1$. While there is nothing wrong geometrically with such systems of coordinates, the existence of the constraint implies that we cannot vary x and y independently in our variational problem (as we we will implicitly do below), and this complicates the analysis of the problem somewhat. So, for simplicity, we will just declare that henceforth we are dealing with unconstrained systems of generalized coordinates.