We can now repeat the derivation of the Euler-Lagrange equations for a general configuration space $\mathcal{C}$. Consider a general path in configuration space given by $\mathbf{q}(t) \in \mathcal{C}$,[7] and assume the existence of a Lagrangian function, $L(\mathbf{q}, \dot{\mathbf{q}})$, such that the action for the path is given by

$$S = \int_{t_0}^{t_1} dt\, L(\mathbf{q}(t), \dot{\mathbf{q}}(t))\,.$$

The variational principle states that, if we fix the initial and final positions in configuration space, that is $\mathbf{q}(t_0) = \mathbf{q}^{(0)}$ and $\mathbf{q}(t_1) = \mathbf{q}^{(1)}$, the path taken by the physical system satisfies

$$\delta S = 0$$

to first order in $\delta\mathbf{q}(t)$. The derivation runs parallel to the one above (here $N \coloneqq \dim(\mathcal{C})$):

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \\
&= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\delta q_i + \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i) \\
&= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i + \frac{d}{dt}\left(\delta q_i \frac{\partial L}{\partial \dot{q}_i}\right) \\
&= \left[\sum_{i=1}^{N} \delta q_i \frac{\partial L}{\partial \dot{q}_i}\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i\,.
\end{aligned}$$

As mentioned above, we are dealing with unconstrained coordinates, meaning that we can vary the q_i independently in configuration space. Since there are $\dim(\mathcal{C})$ independent coordinates, applying the fundamental lemma of the calculus of variations leads to the system of $\dim(\mathcal{C})$ equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad \forall i \in \{1, \ldots, \dim(\mathcal{C})\} \tag{2.2.1}$$

known as the *Euler-Lagrange equations*. I want to emphasize the fact that we have not made any assumptions about the specific choice of coordinate system used in deriving these equations, so the Euler-Lagrange equations are valid in **any** coordinate system.[8]

[7] We know that $\mathbf{q}$ lives in $\mathcal{C}$, by definition. Where does $\dot{\mathbf{q}}$ live? Imagine that at each point in $\mathcal{C}$ we attach a tangent space $T(\mathbf{q})$, the space of all tangent vectors at that point. The vector $\dot{\mathbf{q}}$ is a velocity, so it is a vector in $T(\mathbf{q})$. The total space of all such tangent spaces over all points in $\mathcal{C}$ is known as $T\mathcal{C}$ (the "tangent bundle"). So, if I wanted to be fully precise, I would say that $L\colon T\mathcal{C} \to \mathbb{R}$. While this is the true geometric nature of the Lagrangian function, and the resulting geometric ideas are beautiful to explore, during the course we will take the more pedestrian approach of looking at things locally in $T\mathcal{C}$, where $T\mathcal{C} \approx \mathbb{R}^{\dim(\mathcal{C})} \times \mathbb{R}^{\dim(\mathcal{C})}$. The Lagrangian is then $L\colon \mathbb{R}^{\dim(\mathcal{C})} \times \mathbb{R}^{\dim(\mathcal{C})} \to \mathbb{R}$, that is, a function of two vectors, which we call $\mathbf{q}$ and $\dot{\mathbf{q}}$.

[8] Alternatively, you can derive the Euler-Lagrange equations in any fixed coordinate system, and check that they stay invariant when you change to a different coordinate system, as done in the appendix.

Note 2.2.12

We emphasized in note 2.1.5 above that in the case of systems with one degree of freedom the Lagrangian is a function of the coordinate x (a coordinate in the one-dimensional configurations space) and $\dot{x}$, and these should be treated as independent variables when writing down the Euler-Lagrange equations for the system.

Similarly, for N-dimensional configuration spaces, with generalized coordinates q_i with $i \in \{1, \ldots, N\}$, we have in the Lagrangian formalism

$$\frac{\partial q_i}{\partial \dot{q}_j} = \frac{\partial \dot{q}_i}{\partial q_j} = 0 \tag{2.2.2}$$

and

$$\frac{\partial q_i}{\partial q_j} = \frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.3}$$

Note 2.2.13

We will later on include the possibility of Lagrangians that depend on time explicitly. We indicate this as $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, an example could be $L = \frac{1}{2}m\dot{x}^2 - t^2x^2$.

This is a mild modification of the discussion above, and it does not affect the form of the Euler-Lagrange equations, but there are a couple of things to keep in mind:

1. When taking partial derivatives, t should be taken to be independent from $\mathbf{q}$ and $\dot{\mathbf{q}}$. The reasoning for this is as in note 2.1.5: the Lagrangian is now a function of $2\dim(\mathcal{C}) + 1$ arguments (the generalized coordinates, their velocities, and time), which are unrelated to each other. It is only when we use the Lagrangian to build the action that the parameters become related, but the partial derivatives that appear in the functional variation do not care about this, since they arise in computing the variation of the action under small changes in the path.

   For instance, for $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}t^2x^2$ we have

   $$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad ; \quad \frac{\partial L}{\partial x} = xt^2 \quad ; \quad \frac{\partial L}{\partial t} = -tx^2\,.$$

2. Since in extremizing the action we change the path, but leave the time coordinate untouched, there is no Euler-Lagrange equation associated to t. In the example above there would be a single Euler-Lagrange equation, of the form

   $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = m\ddot{z} + t^2 z = 0\,.$$