§2.3 Lagrangians for classical mechanics

So far we have kept $L(\mathbf{q}, \dot{\mathbf{q}})$ unspecified. How should we choose the Lagrangian in order to reproduce the classical equations of motion? Ultimately, this needs to be decided by experiment, but in problems in classical mechanics there is a very simple prescription, that I will now state. Consider a system with kinetic energy $T(\mathbf{q}, \dot{\mathbf{q}})$ and potential energy $V(\mathbf{q})$. Then the Lagrangian that leads to the right equations of motion is

$$\boxed{L = T - V}$$

Let us see that this gives the right equations of motion in the simple case of a particle moving in three dimensions. The configuration space is $\mathbb{R}^3$, and if we choose Cartesian coordinates x_i (that is, we choose $q_i = x_i$) we have

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

and $V = V(x_1, x_2, x_3)$. Note, in particular, that T depends only on $\dot{x}_i$, and V depends on x_i only. We have three degrees of freedom, so we have three Euler-Lagrange equations, given by

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) \\ &= -\frac{\partial V}{\partial x_i} - m\frac{d}{dt}(\dot{x}_i) \\ &= -\frac{\partial V}{\partial x_i} - m\ddot{x}_i \end{aligned}$$

where we have used that $\frac{\partial V}{\partial \dot{x}_i} = 0$ and $\frac{\partial T}{\partial x_i} = 0$, since x_i and $\dot{x}_i$ are independent variables in the Lagrangian formalism, as we explained above. We can rewrite the equations above in vector notation as

$$m\frac{d^2}{dt^2}(\vec{x}) = -\vec{\nabla}V$$

which is precisely Newton's second law for a conservative force $\vec{F} = -\vec{\nabla}V$.

Example 2.3.1. *The simplest example of the discussion so far is the free particle of mass m moving in d dimensions. Its configuration space is $\mathbb{R}^d$, which we can parametrize using Cartesian coordinates x_i. In these coordinates the kinetic energy is given by*

$$T = \frac{1}{2}m\sum_{i=1}^{d}\dot{x}_i^2$$

and the potential energy V vanishes. This gives a Lagrangian

$$L = T - V = \frac{1}{2}m\sum_{i=1}^{d}\dot{x}_i^2$$

which leads to the d Euler-Lagrange equations of motion

$$m\ddot{x}_i = 0 \quad \forall i \in \{1, \ldots, d\}\,.$$

These equations are solved by the particle moving at constant speed, $x_i = v_i t + b_i$, with v_i, b_i constants.

Example 2.3.2. *Our second example will be a pendulum moving under the influence of gravity. Our conventions will be as in figure 1: we have a mass m attached by a rigid massless rod of length ℓ to a fixed point at the origin. The pendulum can swing on the (x, y) plane. The configuration space of the system is S^1. We choose as a coordinate the angle θ of the rod with the downward vertical axis from the origin, measured counterclockwise. The whole system is affected by gravity, which acts downwards.*

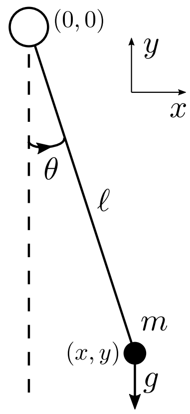

Figure 1: The pendulum discussed in example 2.3.2.

We now need to compute the kinetic and potential energy in terms of θ. The expression of the kinetic energy in the (x, y) coordinates is $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$. In terms of θ we have

$$x = \ell \sin(\theta) \qquad \textit{and} \qquad y = -\ell\cos(\theta)\,.$$

This implies $\dot{x} = \ell\cos(\theta)\dot{\theta}$ and $\dot{y} = \ell\sin(\theta)\dot{\theta}$, so

$$T = \frac{1}{2}m\ell^2\dot{\theta}^2\,.$$

The potential energy, in turn, is (up to an irrelevant additive constant) given by

$$V = mgy = -mg\ell\cos(\theta)$$

leading to the Lagrangian

$$L = T - V = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos(\theta)\,.$$

The corresponding Euler-Lagrange equations are

$$m\ell^2\ddot{\theta} + mg\ell\sin(\theta) = 0$$

or equivalently

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0\,.$$

The exact solution of this system requires using something known as elliptic integrals, but as a simple check of our solution, note that for small angles $\sin(\theta) \approx \theta$*, and the Euler-Lagrange equation reduces to*

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

with solution $\theta(t) = a\sin(\omega t) + b\cos(\omega t)$*, where* $\omega = \sqrt{g/\ell}$*, and* a, b *are arbitrary constants that encode initial conditions. These are the simple oscillatory solutions that one expects close to* $\theta = 0$*.*

Example 2.3.3. *Consider instead a spring with a mass attached to it. The spring is attached on one end to the origin, but it is otherwise free to rotate on the* (x, y) *plane, without friction. In this case we ignore the effect of gravity, and we assume that the spring has vanishing natural length, and constant* κ*. The configuration is shown in figure 2.*

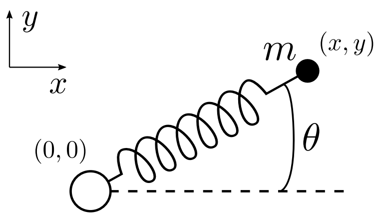

Figure 2: The rotating spring studied in example 2.3.3.

In this case the configuration space is $\mathbb{R}^2$*. It is easiest to solve the Euler-Lagrange equations in Cartesian coordinates. We have the kinetic energy*

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\,.$$

The potential energy is given by the square of the extension of the spring, times the spring constant. We are assuming that the natural length of the spring is 0, so we have that the extension of the spring in $\ell = \sqrt{x^2 + y^2}$*. So the potential energy is*

$$V = \frac{1}{2}\kappa\ell^2 = \frac{1}{2}\kappa(x^2 + y^2)\,.$$

Putting everything together, we find that

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\kappa(x^2 + y^2)\,.$$

The Euler-Lagrange equations split into independent equations for x and y, given by

$$\ddot{x} + \frac{\kappa}{m} x = 0\,,$$
$$\ddot{y} + \frac{\kappa}{m} y = 0\,.$$

The general solution is then simply

$$x(t) = a_x \sin(\omega t) + b_x \cos(\omega t)\,,$$
$$y(t) = a_y \sin(\omega t) + b_y \cos(\omega t)\,,$$

with a_x, a_y, b_x, b_y constants encoding the initial conditions, and $\omega = \sqrt{\kappa/m}$.

Example 2.3.4. *Let us try to solve this last example in polar coordinates r, θ. These are related to Cartesian coordinates by*

$$x = r\cos(\theta)\,,$$
$$y = r\sin(\theta)\,.$$

Taking time derivatives, and using the Chain Rule for time derivatives, we find

$$\dot{x} = \dot{r}\cos(\theta) - r\sin(\theta)\dot{\theta}\,,$$
$$\dot{y} = \dot{r}\sin(\theta) + r\cos(\theta)\dot{\theta}\,.$$

A little bit of algebra then shows that

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2)\,.$$

On the other hand, the potential energy is simpler. We have

$$V = \frac{1}{2}\kappa(x^2 + y^2) = \frac{1}{2}\kappa r^2\,.$$

We thus find that the Lagrangian in polar coordinates is

$$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}\kappa r^2\,.$$

Let us write the Euler-Lagrange equations. For the coordinate r we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 + \kappa r = 0$$

while for the θ coordinate we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0\,.$$

This equation is quite remarkable: it tells us that there is a conserved quantity in this system, given by $mr^2\dot{\theta}$*. This was not obvious at all in the Cartesian formulation of the problem,*[9] *but it follows immediately in polar coordinates, since the Lagrangian does not depend on* θ*, only on* $\dot{\theta}$*, and accordingly* $\partial L/\partial\theta = 0$*. We can use this knowledge to simplify the problem. Define*

$$J \coloneqq mr^2\dot{\theta}\,.$$

This is a constant of motion, so on any given classical trajectory it is simply a real number fixed by initial conditions. We can use this knowledge to simplify the Euler-Lagrange equation for r*, which after replacing* $\dot{\theta} = J/mr^2$ *becomes an equation purely in terms of* r*:*

$$m\ddot{r} - mr\left(\frac{J}{mr^2}\right)^2 + \kappa r = m\ddot{r} - \frac{J^2}{mr^3} + \kappa r = 0\,.$$

[9] Once we know that the conserved charged is there, it is not difficult to find its expression in Cartesian coordinates: we have $mr^2\dot{\theta} = m(x\dot{y} - y\dot{x})$.