§2.4 Ignorable coordinates and conservation of generalised momenta

It is useful to formalize what we just saw happen in example 2.3.3.

Definition 2.4.1. Given a set $\{q_1, \ldots, q_N\}$ of generalized coordinates, we say that a specific coordinate q_i in the set is *ignorable* if the Lagrangian function, expressed in these generalised coordinates, does not depend on q_i. That is, a coordinate is ignorable iff

$$\frac{\partial L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N)}{\partial q_i} = 0\,.$$

Definition 2.4.2. The *generalized momentum* p_i associated to a generalized coordinate is

$$p_i \coloneqq \frac{\partial L}{\partial \dot{q}_i}\,.$$

With these two definitions in place we have

Proposition 2.4.3. *The generalized momentum associated to an ignorable coordinate is conserved.*

Proof. This follows immediately from the Euler-Lagrange equation for the ignorable coordinate. Denoting the ignorable coordinate q_i and its associated generalized momentum p_i, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \frac{dp_i}{dt} - 0 = \frac{dp_i}{dt} = 0\,.$$

□

Example 2.4.4. *We already found a ignorable coordinate in example 2.3.3. We have that θ was ignorable, and it associated generalized momentum is*

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\,.$$

Example 2.4.5. *An even simpler example is the free particle moving in d dimensions. In Cartesian coordinates we have*

$$L = T - V = \frac{1}{2}m\sum_{i=1}^{d}\dot{x}_i^2\,,$$

so every coordinate is ignorable. The associated generalized momenta are

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i\,.$$

In this case conservation of generalized momenta is simply conservation of linear momentum.

Example 2.4.6. *Let us look again to the free particle, but this time in two dimensions ($d = 2$), and in polar coordinates. We have*

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\,.$$

We have that θ is ignorable. The associated conserved generalized momentum is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\,.$$

You might recognize this as the angular momentum of the particle (that is, linear momentum $\times$ position vector), which should indeed be conserved for the free particle.