§3 Symmetries, Noether's theorem and conservation laws

§3.1 Ordinary symmetries

Our discussion of ignorable coordinates hints at a connection between symmetries and conservation laws: the fact that the Lagrangian does not depend on q_i can be rephrased as the statement that the Lagrangian is invariant under the transformation $q_i \to q_i + \epsilon a_i$, with ϵa_i an arbitrary constant shift. (We will define all these concepts more carefully momentarily.) And we saw that whenever this happens, there is a conserved quantity, the generalized momentum p_i.

This result is somewhat unsatisfactory, in that we can only understand the appearance of the conserved charges in carefully chosen coordinate systems. And, as we saw in the example of the free particle above, we might need to patch together results in different coordinate systems in order to access all the conserved charges in the system.

Noether's theorem fixes these deficiencies, providing a coordinate-independent connection between symmetries and conservation laws. Before we get to the theorem itself, we will need some preliminary results and definitions.

Definition 3.1.1. Consider a uniparametric family of smooth maps $\varphi(\epsilon)\colon \mathcal{C} \to \mathcal{C}$ from configuration space to itself, with the property that $\varphi(0)$ is the identity map. We call this family of maps a *transformation depending on* ϵ. In any given coordinate system we can write the transformation as

$$q_i \to \phi_i(q_1, \ldots, q_N, \epsilon)$$

with ϕ_i a set of $N \coloneqq \dim(\mathcal{C})$ functions representing the finite transformation in the given coordinate system. We take the change in velocities to be

$$\dot{q}_i \to \frac{d}{dt}\phi_i \,.$$

Note 3.1.2

At the level of the Lagrangian we treat q_i and $\dot{q}_i$ as independent variables, so it is not automatic that the transformation of the velocities $\dot{q}_i$ is as given. One should take the prescription $\dot{q}_i \to \frac{d}{dt}\phi_i$ as part of the definition above.

Remark 3.1.3. A word on notation: when it is clear from the context which transformation we are talking about, we often write q_i' instead of $\phi_i(\mathbf{q}, \epsilon)$. That is, we often write

$$q_i \to q_i' = \ldots$$

where the omitted terms are some function of q_i and ϵ.

Definition 3.1.4. The *generator* of φ is

$$\frac{d\varphi(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} := \lim_{\epsilon\to 0}\frac{\varphi(\epsilon)-\varphi(0)}{\epsilon}\,.$$

In any given coordinate system we have

$$q_i \to \phi_i(\mathbf{q},\epsilon) = q_i + \epsilon a_i(\mathbf{q}) + \mathcal{O}(\epsilon^2)$$

where

$$a_i = \frac{\partial\phi_i(\mathbf{q},\epsilon)}{\partial\epsilon}\Big|_{\epsilon=0}$$

is a function of the generalized coordinates. So, in coordinates, the generator of the transformation is a_i. Similarly, for the velocities we have

$$\dot q_i \to \dot q_i + \epsilon \dot a_i(q_1,\dots,q_N,\dot q_1,\dots,\dot q_N) + \mathcal{O}(\epsilon^2)$$

generated by $\dot a_i$.

Example 3.1.5. *A particle moving in $\mathbb{R}^d$ can be described in Cartesian coordinates x_i. The transformation associated to translations of the origin of coordinates in the first direction is $x_1 \to x_1+\epsilon$, with the other coordinates constant. So we have that shifts of the coordinate system in the x_1 direction are generated by*

$$a_i = \begin{cases} 1 & \textit{for } i=1\,, \\ 0 & \textit{otherwise} \end{cases}$$

and $\dot a_i = 0$.

Example 3.1.6. *Say that we have a particle moving in two dimensions, and we want to consider the finite transformations given by rotations around the origin. In Cartesian coordinates we have*

$$\begin{aligned} x &\to x\cos(\epsilon) - y\sin(\epsilon) \\ y &\to x\sin(\epsilon) + y\cos(\epsilon)\,. \end{aligned}$$

In order to find the generators, we can derive the associated infinitesimal transformations by using the expansions $\sin(\epsilon) = \epsilon + \mathcal{O}(\epsilon^3)$ and $\cos(\epsilon) = 1 + \mathcal{O}(\epsilon^2)$. We find

$$\begin{aligned} x &\to x - y\epsilon + \mathcal{O}(\epsilon^2) \\ y &\to y + x\epsilon + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

This implies that the transformation is generated in Cartesian coordinates by

$$a_x = -y \qquad ; \qquad a_y = x \qquad ; \qquad \dot a_x = -\dot y \qquad ; \qquad \dot a_y = \dot x\,.$$

Lemma 3.1.7. *The equations of motion do not change if we modify the Lagrangian by addition of a total derivative of a function of coordinates and time. That is,*

$$L \to L + \frac{dF(q_1, \ldots, q_N, t)}{dt}$$

does not affect the equations of motion.

Proof. Since the term that we add is a total time derivative, the effect on the action is

$$S = \int_{t_0}^{t_1} dt\, L \to S' = S + F(q_1(t_1), \ldots, q_N(t_1), t_1) - F(q_1(t_0), \ldots, q_N(t_0), t_0)\,. \tag{3.1.1}$$

Now, recall that the variational principle tells us that the equations of motion are obtained by imposing that δS vanishes to first order in $\delta q_i(t)$, keeping the q_i fixed at the endpoints of the path. This implies that in the variational problem both $F(q_1(t_0), \ldots, q_N(t_0), t_0)$ and $F(q_1(t_1), \ldots, q_N(t_1), t_1)$ are kept fixed. So

$$\begin{aligned}
\delta S' &= S'[\mathbf{q} + \delta\mathbf{q}] - S'[\mathbf{q}] \\
&= S[\mathbf{q} + \delta\mathbf{q}] + F(q_1(t_1), \ldots, q_N(t_1), t_1) - F(q_1(t_0), \ldots, q_N(t_0), t_0) \\
&\quad - (S[\mathbf{q}] + F(q_1(t_1), \ldots, q_N(t_1), t_1) - F(q_1(t_0), \ldots, q_N(t_0), t_0)) \\
&= S[\delta\mathbf{q}] - S[\mathbf{q}] = \delta S\,.
\end{aligned}$$

We learn that the addition of $\frac{dF}{dt}$ to the Lagrangian does not affect the variation of the action in the variational problem, so it cannot affect the equations of motion. $\square$