This result motivates the following definition:

Definition 3.1.8. A transformation $\varphi(\epsilon)$ is a *symmetry* if, to first order in ϵ, there exists some function $F(\mathbf{q}, t)$ such that the change in the Lagrangian is a total time derivative of $F(\mathbf{q}, t)$:

$$L \to L' = L(\phi(q_1, \epsilon), \ldots, \phi(q_N, \epsilon)) = L + \epsilon \frac{dF(q_1, \ldots, q_N, t)}{dt} + \mathcal{O}(\epsilon^2)\,.$$

Remark 3.1.9. I emphasize that $F(\mathbf{q}, t)$ is only defined up to a constant: if some $F(\mathbf{q}, t)$ exists such that

$$L' = L + \epsilon \frac{dF(\mathbf{q}, t)}{dt} + \mathcal{O}(\epsilon^2)$$

any other $F'(\mathbf{q}, t) = F(\mathbf{q}, t) + c$ with c is a constant will also satisfy the same equation. The specific choice of c is arbitrary, and any choice will lead to correct results. In what follows I will simply pick a convenient representative $F(\mathbf{q}, t)$ — for instance $F(\mathbf{q}, t) = 0$ whenever this is possible.

Example 3.1.10. *Whenever we have an ignorable coordinate q_i, the symmetry associated to shifting it by constants, $q_i \to q_i + c_i$, is clearly a symmetry, since by definition the coordinate does not appear in the Lagrangian, and $\dot{q}_i$ stays invariant. So in this case F can be chosen to be 0.*

As an example, consider the example of the rotating spring discussed in example 2.3.3. In polar coordinates (r, θ), we have

$$L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}\kappa r^2\,.$$

In this case the θ coordinate is ignorable, so the associated shift $\theta \to \theta + \epsilon$ is a symmetry. The generators of the symmetry are

$$a_r = 0 \quad ; \quad a_\theta = 1 \quad ; \quad \dot{a}_r = 0 \quad ; \quad \dot{a}_\theta = 0\,.$$

Example 3.1.11. *Let us study the same system as in the previous example, but now in Cartesian coordinates. We have*

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}\kappa(x^2 + y^2)\,.$$

The transformation $\theta \to \theta + \epsilon$ is a rotation around the origin. Whenever $\epsilon \ll 1$, we have

$$\begin{aligned} x \to x' &= x - \epsilon y + \mathcal{O}(\epsilon^2) \\ y \to y' &= y + \epsilon x + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

as we argued in example 3.1.6. And accordingly, for the time derivatives we have

$$\begin{aligned} \dot{x} \to \dot{x}' &= \dot{x} - \dot{y}\epsilon + \mathcal{O}(\epsilon^2) \\ \dot{y} \to \dot{y}' &= \dot{y} + \dot{x}\epsilon + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

Note that this transformations imply that

$$x^2 + y^2 \to x'^2 + y'^2 = (x + \epsilon y)^2 + (y - \epsilon x)^2 = x^2 + y^2 + \mathcal{O}(\epsilon^2)$$

and similarly that

$$\dot{x}^2 + \dot{y}^2 \to \dot{x}'^2 + \dot{y}'^2 = \dot{x}^2 + \dot{y}^2 + \mathcal{O}(\epsilon^2)$$

The action of the symmetry on the Lagrangian is then, to first order in ϵ*:*

$$L \to L' = L(x', y', \dot{x}', \dot{y}') = L + \mathcal{O}(\epsilon^2)$$

so we also see in this coordinate system that the rotation is a symmetry.

Note that this argument generalizes straightforwardly for any Lagrangian of the form

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2)$$

with $V(r)$ *an analytic function of* r*, since in this case*

$$V(x^2 + y^2 + \mathcal{O}(\epsilon^2)) = V(x^2 + y^2) + \mathcal{O}(\epsilon^2)\,.$$

Example 3.1.12. *Consider a system with Lagrangian*

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - y\dot{x} - \frac{1}{2}x^2\,,$$

and a transformation generated by

$$\begin{aligned} x &\to x' = x\,, \\ y &\to y' = y + \epsilon\,. \end{aligned}$$

Then $\dot{x}' = \dot{x}$ *and* $\dot{y}' = \dot{y}$ *and*

$$\delta L = L(x', y', \dot{x}', \dot{y}') - L(x, y, \dot{x}, \dot{y}) = -y'\dot{x}' + y\dot{x} = -\epsilon\dot{x}\,.$$

So this is also a symmetry, this time with $F = -x$*.*

Note 3.1.13

It is important to notice that the definition of *symmetry* above does not involve the equations of motion: the Lagrangian must stay invariant (up to a total derivative) **without** using the equations of motion. That is, the Lagrangian must be invariant also for those paths in configuration space that do not extremize S.