We are finally in a position to state and prove Noether's theorem.

Theorem 3.1.14 (Noether)**.** *Consider a transformation generated by $a_i(q_1, \ldots, q_N)$ (in a given set of generalized coordinates), such that*

$$L \to L + \epsilon \frac{dF(q_1, \ldots, q_N, t)}{dt} + \mathcal{O}(\epsilon^2)\,,$$

so that it is a symmetry. Then

$$\boxed{Q := \left(\sum_{i=1}^{N} a_i \frac{\partial L}{\partial \dot{q}_i}\right) - F}$$

is conserved (that is, $\frac{dQ}{dt} = 0$). The conserved quantity Q is known as the Noether charge*.*

Proof. I will start by giving the intuitive idea behind the proof. Recall that physical trajectories $q_i(t)$ are those that satisfy $\delta S = 0$ to first order in $\delta q_i(t)$, keeping the endpoints $q_i(t_0)$ and $q_i(t_1)$ fixed. A general transformation acts as $q_i(t) \to q(t) + \epsilon a_i(q, t)$, but crucially it does not necessarily keep the endpoints $q_i(t_0)$ and $q_i(t_1)$ fixed. So the action of a physical path can change to first order in ϵ under a generic transformation. But it does so in a fairly localised way: only the behaviour near the endpoints of the path, at t_0 and t_1, can contribute to δS. If the transformation is furthermore a symmetry, we can compute δS (to first order in ϵ) in a second way, as a function of quantities at t_0 and t_1 only, using our result (3.1.1) above. Equating the result of both approaches leads to Noether's theorem.

In detail, this goes a follows. We want to understand the variation of the action under the transformation

$$q_i \to q_i + \delta q_i = q_i + \epsilon a_i$$

in two different ways. On one hand, as for any other variation of the path, we can Taylor expand to obtain

$$\delta S = \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\epsilon a_i \frac{\partial L}{\partial q_i} + \epsilon \dot{a}_i \frac{\partial L}{\partial \dot{q}_i}\right) + \mathcal{O}(\epsilon^2)$$

which becomes, using the Euler-Lagrange equations

$$\begin{aligned}
\delta S &= \int_{t_0}^{t_1} dt \sum_{i=1}^{N} \left(\epsilon a_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \epsilon \dot{a}_i \frac{\partial L}{\partial \dot{q}_i}\right) + \mathcal{O}(\epsilon^2) \\
&= \int_{t_0}^{t_1} dt\, \epsilon \frac{d}{dt}\left(\sum_{i=1}^{N} a_i \frac{\partial L}{\partial \dot{q}_i}\right) + \mathcal{O}(\epsilon^2) \\
&= \epsilon \left[\sum_{i=1}^{N} a_i \frac{\partial L}{\partial \dot{q}_i}\right]_{t_0}^{t_1} + \mathcal{O}(\epsilon^2)\,.
\end{aligned}$$

Note that we have used the Euler-Lagrange equations of motion in going from the first to the second line, so the result will only be valid along the path that satisfies the equations of motion.

On the other hand, using the fact that the variation is a symmetry, we have

$$\begin{aligned}\delta S &= S[\mathbf{q}+\delta\mathbf{q}] - S[\mathbf{q}] \\ &= \int_{t_0}^{t_1} dt \left(\left(L + \epsilon\frac{dF}{dt} + \mathcal{O}(\epsilon^2)\right) - L\right) \\ &= \epsilon\,[F]_{t_0}^{t_1} + \mathcal{O}(\epsilon^2)\,.\end{aligned}$$

Equation both results, we immediately obtain that $Q(t_1) = Q(t_0)$. Since the choice of t_0 and t_1 is arbitrary, the result now follows easily: choose $t_1 = t_0 + \epsilon$. We have

$$Q(t_1) - Q(t_0) = Q(t_0+\epsilon) - Q(t_0) = \epsilon\frac{dQ}{dt} + \mathcal{O}(\epsilon^2) = 0$$

so $\frac{dQ}{dt} = 0$. □

Example 3.1.15. *Whenever the coordinate q_i is ignorable, we have a symmetry (with $f = 0$) generated by $q_i \to q_i + \epsilon$, leaving the other coordinates constant. That is,*

$$a_k = \delta_{ik} := \begin{cases} 1 & \text{if } i = k\,. \\ 0 & \text{otherwise.}\end{cases}$$

The corresponding Noether charge is then

$$Q = \sum_{k=1}^{N} a_k \frac{\partial L}{\partial \dot{q}_i} = \sum_{k=1}^{N} \delta_{ki}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i}$$

as expected.

Example 3.1.16. *Let us come back to the conservation of angular momentum in rotationally symmetric systems, expressed in Cartesian coordinates. Assume that we have a system with Lagrangian*

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x^2+y^2)\,.$$

We saw in example 3.1.11 that rotations around the origin, which are generated by

$$a_x = -y \qquad ; \qquad a_y = x\,,$$

are a symmetry of the system with $F = 0$.

Noether's theorem then tells us that the associated charge is

$$Q = a_x\frac{\partial L}{\partial \dot{x}} + a_y\frac{\partial L}{\partial \dot{y}} = m(-y\dot{x} + x\dot{y})\,.$$

It is a simple exercise to show that this is indeed equal to $mr^2\dot{\theta}$.

Example 3.1.17. *Finally, let us revisit example 3.1.12. We have a Lagrangian*

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - y\dot{x} - \frac{1}{2}x^2\,,$$

and a transformation generated by

$$\begin{aligned} x &\to x' = x\,, \\ y &\to y' = y + \epsilon\,. \end{aligned}$$

That is, $a_x = 0$ and $a_y = 1$. We found in example 3.1.12 that this transformation is a symmetry with $F = -x$. The associated Noether charge is

$$Q = a_x\frac{\partial L}{\partial \dot{x}} + a_y\frac{\partial L}{\partial \dot{y}} - F = m\dot{y} + x\,.$$

We can check that this is conserved from the equations of motion, which are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= m\ddot{x} - \dot{y} + x = 0\,, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= m\ddot{y} + \dot{x} = 0\,. \end{aligned}$$

Note in particular that the second equation is precisely $\frac{dQ}{dt} = 0$.