§3.2 Non-canonical kinetic terms

Finally, we consider configurations with non-canonical kinetic terms of the form

$$L = \frac{1}{2}\sum_{i,j} \mathsf{B}_{ij}(q)\dot{q}_i\dot{q}_j - V(q)\,. \tag{3.2.1}$$

We still obtain a linear differential operator if we restrict $\mathsf{B}(q) \to \mathsf{B}(0)$. Physically, this corresponds to considering oscillations with not too much kinetic energy, which makes sense if we want to stay at the minimum. The resulting equations of motion are

$$\mathsf{B}\ddot{\mathbf{q}} + \mathsf{A}\mathbf{q} = 0 \tag{3.2.2}$$

where we have defined $\mathsf{B} := \mathsf{B}(0)$, a constant matrix. B generally does not have zero eigenvalues (since this would correspond to generalised coordinates without a kinetic term), so we assume no zero eigenvalues. This implies that $\det(\mathsf{B}) \neq 0$, and so B^{-1} exists. We then have an equivalent set of equations

$$\ddot{\mathbf{q}} + \mathsf{B}^{-1}\mathsf{A}\mathbf{q} = 0 \tag{3.2.3}$$

which reduces to the case we have already studied if you define $\mathsf{C} := \mathsf{B}^{-1}\mathsf{A}$.

There is one small subtlety that needs to be mentioned here: the fact that A was symmetric was quite important in our discussion above, since it ensured that its eigenvalues were real, but in general $\mathsf{B}^{-1}\mathsf{A}$ will not be symmetric, even if both B^{-1} and A separately are. Let us assume that A is positive definite: that is, all its eigenvalues are positive. Then it exists a symmetric matrix $\mathsf{A}^{\frac{1}{2}}$ such that $(\mathsf{A}^{\frac{1}{2}})^2 = \mathsf{A}$.[10] We can use this matrix to rewrite

$$\mathsf{C} := \mathsf{B}^{-1}\mathsf{A} = \mathsf{A}^{-\frac{1}{2}}\left(\mathsf{A}^{\frac{1}{2}}\mathsf{B}^{-1}\mathsf{A}^{\frac{1}{2}}\right)\mathsf{A}^{\frac{1}{2}}$$

so we find that C is similar (in the sense of similarity transformations of matrices) to $\mathsf{A}^{\frac{1}{2}}\mathsf{B}^{-1}\mathsf{A}^{\frac{1}{2}}$. This matrix is manifestly symmetric, so its eigenvalues are real. Since similar matrices have the same eigenvalues, the eigenvalues of C will be real too. It is straightforward to check that if $\mathbf{v}$ is an eigenvector of $\mathsf{A}^{\frac{1}{2}}\mathsf{B}^{-1}\mathsf{A}^{\frac{1}{2}}$ with eigenvalue λ, then $\mathsf{A}^{-\frac{1}{2}}\mathbf{v}$ will be an eigenvector of C with the same eigenvalue. So, in practice, we can simply compute the eigenvalues and eigenvectors of C, and proceed as we did above.

[10] The simplest way to prove this is to note that A is real symmetric, and thus diagonalisable by an orthogonal transformation O as $\mathsf{A} = ODO^t$. We can then define $\mathsf{A}^{\frac{1}{2}} = OD^{\frac{1}{2}}O^t$.