§3.2 Energy conservation

Conservation of energy can be understood in a way quite similar to what we have seen: energy can be defined as the Noether charge associated with time translations. The derivation is quite similar to the one above, but with some small (but crucial) differences needed in order to take into account the fact that the time coordinate "t" is treated specially in the Lagrangian formalism.

Let us consider the possibility that the Lagrangian depends explicitly on time. That is, we promote the Lagrangian L to a function of the generalized coordinates $q_i(t)$, their associated velocities $\dot{q}_i(t)$, and time itself. We write this as $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. The expression of the action is now

$$S = \int_{t_0}^{t_1} L(q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots, \dot{q}_N(t), t)\, dt\,.$$

It is not difficult to see that the Euler-Lagrange equations do not change if we do this.[10]

Definition 3.2.1. Given a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, we defined the *energy* to be

$$\boxed{E := \left(\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L}$$

Theorem 3.2.2. *Along a path* $\mathbf{q}(t)$ *satisfying the equations of motion, we have*

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}\,.$$

In particular, the energy is conserved if and only if the Lagrangian does not depend explicitly on time.

Remark 3.2.3. In this theorem $\frac{\partial L}{\partial t}$ denotes taking the derivative of the Lagrangian with respect to time, keeping $\mathbf{q}$ and $\dot{\mathbf{q}}$ fixed. See note 2.2.13 for a further discussion of this point.

Elementary proof. It is easy to verify directly, by taking the time derivative of the definition of energy, that the theorem holds. The calculation goes as follows. If we take the time derivative of the energy, we have (from definition 3.2.1)

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left(\left(\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L\right)\\ &= \left(\sum_{i=1}^{N} \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right) - \frac{dL}{dt}\end{aligned}$$

[10] I leave this as an exercise. All you need to do is to convince yourself that our derivation of the Euler-Lagrange equations, above equation (2.2.1), is not modified if the Lagrangian includes an explicit dependence on time.

Using the Euler-Lagrange equations, this becomes

$$\frac{dE}{dt} = \left(\sum_{i=1}^{N} \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial L}{\partial q_i} \right) - \frac{dL}{dt} \, .$$

On the other hand, from the Chain Rule, we have

$$\frac{dL}{dt} = \left(\sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial L}{\partial q_i} \dot{q}_i \right) + \frac{\partial L}{\partial t} \, .$$

The result now follows from substitution. $\square$