

Alternative proof. Here I will present a less straightforward but (in my opinion) more illuminating proof, closer in spirit to the one we used in proving Noether's theorem.

i This alternative proof is **not examinable**. **i**

Imagine that we take a path $\mathbf{q}(t)$ satisfying the equations of motion, and we displace it to a new path $\mathbf{q}'(t) = \mathbf{q}(t - \epsilon)$. That is, we move the whole path slightly forward in time, keeping its shape. We have

$$\begin{aligned} S' &= \int_{t_0}^{t_1} dt L(q'_1(t), \dots, q'_N(t), \dot{q}'_1(t), \dots, \dot{q}'_N(t), t) \\ &= \int_{t_0}^{t_1} dt L(q_1(t - \epsilon), \dots, q_N(t - \epsilon), \dot{q}_1(t - \epsilon), \dots, \dot{q}_N(t - \epsilon), t). \end{aligned}$$

We can compute this expression in two different ways. First, by the Chain Rule, we have that

$$\begin{aligned} L(q_1(t - \epsilon), \dots, q_N(t - \epsilon), \dot{q}_1(t - \epsilon), \dots, \dot{q}_N(t - \epsilon), t) = \\ L(q_1(t), \dots, q_N(t), \dot{q}_1(t), \dots, \dot{q}_N(t), t) - \epsilon \left(\sum_{i=1}^N \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) + \mathcal{O}(\epsilon^2). \end{aligned}$$

Using the Euler-Lagrange equations of motion, we can write this as

$$\begin{aligned} \sum_{i=1}^N \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i &= \sum_{i=1}^N \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \\ &= \frac{d}{dt} \left(\sum_{i=1}^N \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right). \end{aligned}$$

Substituting these expressions into the action, we have just proven that

$$S' = S - \epsilon \left[\sum_{i=1}^N \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right]_{t_0}^{t_1} + \mathcal{O}(\epsilon^2).$$

On the other hand, introducing a new variable $t' = t - \epsilon$, we can write

$$\begin{aligned} S' &= \int_{t_0}^{t_1} dt L(q_1(t - \epsilon), \dots, q_N(t - \epsilon), \dot{q}_1(t - \epsilon), \dots, \dot{q}_N(t - \epsilon), t) \\ &= \int_{t_0 - \epsilon}^{t_1 - \epsilon} dt' L(q_1(t'), \dots, q_N(t'), \dot{q}_1(t'), \dots, \dot{q}_N(t'), t' + \epsilon). \end{aligned}$$

We can expand this as a series in ϵ using Leibniz's rule (see equation (A.0.1) in the appendix for a reminder), to get:

$$\begin{aligned}
S' &= S + \epsilon \left. \frac{dS'}{d\epsilon} \right|_{\epsilon=0} + \mathcal{O}(\epsilon^2) \\
&= S - \epsilon L(q_1(t_1), \dots, q_N(t_1), \dot{q}_1(t_1), \dots, \dot{q}_N(t_1), t_1) \\
&\quad + \epsilon L(q_1(t_0), \dots, q_N(t_0), \dot{q}_1(t_0), \dots, \dot{q}_N(t_0), t_0) \\
&\quad + \epsilon \left[\int_{t_0-\epsilon}^{t_1-\epsilon} dt' \frac{\partial L(q_1(t'), \dots, q_N(t'), \dot{q}_1(t'), \dots, \dot{q}_N(t'), t' + \epsilon)}{\partial \epsilon} \right]_{\epsilon=0} \\
&\quad + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Now we note that that, by the Chain Rule, we have

$$\frac{\partial L(q_1(t'), \dots, q_N(t'), \dot{q}_1(t'), \dots, \dot{q}_N(t'), t' + \epsilon)}{\partial \epsilon} = \frac{\partial L(q_1(t'), \dots, q_N(t'), \dot{q}_1(t'), \dots, \dot{q}_N(t'), t' + \epsilon)}{\partial t'}$$

so

$$\begin{aligned}
&\left[\int_{t_0-\epsilon}^{t_1-\epsilon} dt' \frac{\partial L(q_1(t'), \dots, q_N(t'), \dot{q}_1(t'), \dots, \dot{q}_N(t'), t' + \epsilon)}{\partial \epsilon} \right]_{\epsilon=0} \\
&= \int_{t_0}^{t_1} dt \frac{\partial L(q_1(t), \dots, q_N(t), \dot{q}_1(t), \dots, \dot{q}_N(t), t)}{\partial t}.
\end{aligned}$$

The theorem now follows from equating the two expressions for S' that we found. \square