Alternative proof. Here I will present a less straightforward but (in my opinion) more illuminating proof, closer in spirit to the one we used in proving Noether's theorem.

> ⓘ   *This alternative proof is **not examinable**.*   ⓘ

Imagine that we take a path $\mathbf{q}(t)$ satisfying the equations of motion, and we displace it to a new path $\mathbf{q}'(t) = \mathbf{q}(t - \epsilon)$. That is, we move the whole path slightly forward in time, keeping its shape. We have

$$
S' = \int_{t_0}^{t_1} dt\, L(q_1'(t), \ldots, q_N'(t), \dot{q}_1'(t), \ldots, \dot{q}_N'(t), t)
$$

$$
= \int_{t_0}^{t_1} dt\, L(q_1(t - \epsilon), \ldots, q_N(t - \epsilon), \dot{q}_1(t - \epsilon), \ldots, \dot{q}_N(t - \epsilon), t)\,.
$$

We can compute this expression in two different ways. First, by the Chain Rule, we have that

$$
L(q_1(t - \epsilon), \ldots, q_N(t - \epsilon), \dot{q}_1(t - \epsilon), \ldots, \dot{q}_N(t - \epsilon), t) =
$$

$$
L(q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots, \dot{q}_N(t), t) - \epsilon \left(\sum_{i=1}^{N} \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) + \mathcal{O}(\epsilon^2)\,.
$$

Using the Euler-Lagrange equations of motion, we can write this as

$$
\sum_{i=1}^{N} \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i = \sum_{i=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i
$$

$$
= \frac{d}{dt}\left(\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right)\,.
$$

Substituting these expressions into the action, we have just proven that

$$
S' = S - \epsilon \left[\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right]_{t_0}^{t_1} + \mathcal{O}(\epsilon^2)\,.
$$

On the other hand, introducing a new variable $t' = t - \epsilon$, we can write

$$
S' = \int_{t_0}^{t_1} dt\, L(q_1(t - \epsilon), \ldots, q_N(t - \epsilon), \dot{q}_1(t - \epsilon), \ldots, \dot{q}_N(t - \epsilon), t)
$$

$$
= \int_{t_0 - \epsilon}^{t_1 - \epsilon} dt'\, L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t' + \epsilon)\,.
$$

We can expand this as a series in ϵ using Leibniz's rule (see equation (A.0.1) in the appendix for a reminder), to get:

$$
\begin{aligned}
S' &= S + \epsilon \frac{dS'}{d\epsilon}\Big|_{\epsilon=0} + \mathcal{O}(\epsilon^2) \\
&= S - \epsilon L(q_1(t_1), \ldots, q_N(t_1), \dot{q}_1(t_1), \ldots, \dot{q}_N(t_1), t_1) \\
&\quad + \epsilon L(q_1(t_0), \ldots, q_N(t_0), \dot{q}_1(t_0), \ldots, \dot{q}_N(t_0), t_0) \\
&\quad + \epsilon \left[\int_{t_0-\epsilon}^{t_1-\epsilon} dt' \, \frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial \epsilon} \right]_{\epsilon=0} \\
&\quad + \mathcal{O}(\epsilon^2)
\end{aligned}
$$

Now we note that that, by the Chain Rule, we have

$$
\frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial \epsilon} = \frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial t'}
$$

so

$$
\begin{aligned}
\left[\int_{t_0-\epsilon}^{t_1-\epsilon} dt' \, \frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial \epsilon} \right]_{\epsilon=0} & \\
= \int_{t_0}^{t_1} dt \, \frac{\partial L(q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots, \dot{q}_N(t), t)}{\partial t} &.
\end{aligned}
$$

The theorem now follows from equating the two expressions for S' that we found. $\square$