Note 3.2.4

It is not obvious that the quantity E that is conserved if $\frac{\partial L}{\partial t} = 0$ is what is usually known as "energy" in classical mechanics. But this is easy to verify. Assume that we have a particle with Lagrangian

$$L = T(\dot{x}_1, \ldots, \dot{x}_N) - V(x_1, \ldots, x_N)$$

with $T(\dot{x}_1, \ldots, \dot{x}_N) = \frac{1}{2}m(\dot{x}_1^2 + \ldots + \dot{x}_d^2)$, as we often do in classical mechanics. Then applying the definition 3.2.1 above one easily finds the expected relation

$$E = T + V\,.$$

The result holds more generally. Consider a Lagrangian of the form

$$L = \underbrace{\left(\sum_{i,j=1}^{N} K_{ij}(q_1, \ldots, q_N)\dot{q}_i\dot{q}_j\right)}_{T(q,\dot{q})} - V(q)$$

with the $K_{ij}(q)$ and $V(q)$ arbitrary functions on configuration space $\mathcal{C}$. Then it is easy to verify that

$$E = T + V\,.$$

Example 3.2.5. *Say that we have a spring that becomes weaker with time, with a spring constant $\kappa(t) = e^{-t}$. A mass attached to the spring can then be described by a Lagrangian*

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\kappa(t)x^2\,.$$

The resulting equation of motion is

$$m\ddot{x} + \kappa(t)x = 0\,.$$

The energy of the system is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\kappa(t)x^2\,.$$

Since the Lagrangian depends explicitly on time, we expect that energy is not conserved. And indeed:

$$\begin{aligned}\frac{dE}{dt} &= m\dot{x}\ddot{x} + \kappa(t)x\dot{x} + \frac{1}{2}x^2\frac{d\kappa(t)}{dt}\\ &= \dot{x}(m\ddot{x} + \kappa(t)x) + \frac{1}{2}x^2\frac{d\kappa(t)}{dt}\\ &= \frac{1}{2}x^2\frac{d\kappa(t)}{dt}\end{aligned}$$

where in the last step we have used the equation of motion. On the other hand

$$\frac{\partial L}{\partial t} = -\frac{1}{2}x^2 \frac{d\kappa(t)}{dt}$$

since time appears explicitly only in $\kappa(t)$. So we have verified that

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}\,.$$

Example 3.2.6. *Note that our definition 3.2.1 for the energy does not require the Lagrangian to have the specific form $L = T - V$. Consider for instance the Lagrangian*

$$L = -m\left(\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\right)\,.$$

(This specific Lagrangian is in fact fairly important, as it describes the motion of a particle in $\mathbb{R}^3$ in special relativity.) Definition 3.2.1 gives

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}} - L\,.$$

We have

$$\dot{x}\frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}^2}{\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}},$$

and similarly for $\dot{y}$ and $\dot{z}$. Putting everything together we find

$$\begin{aligned} E &= \frac{m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}} + m\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} \\ &= \frac{m}{\sqrt{1 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}}\,. \end{aligned}$$