§4 Normal modes

So far we have studied the Euler-Lagrange equations abstractly, but we have not spent much effort actually trying to solve them, except on some fairly elementary examples. The reason for this is simple: in most cases we cannot solve the equations in closed form. Even if we can, it is rarely the case that the answer can be written in terms of elementary functions. Recall, for instance, example 2.3.2 above, where we discussed the pendulum. We found that the Euler-Lagrange equations of motion were of the form

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0\,.$$

This equation can be solved in closed form, in terms of a class of special functions known as "elliptic functions", but the solution is relatively involved, and not particularly illuminating for our current purposes. Rather than insisting in solving the problem exactly from the outset, it is often illuminating to instead try to understand what the system does for small displacements away from equilibrium. That is, for small values of θ. In this regime we have that $\sin(\theta) \approx \theta$, and the equation of motion becomes

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

which can be solved straightforwardly to give

$$\theta = a\cos(\omega t) + b\sin(\omega t)$$

with $\omega = \sqrt{\frac{g}{\ell}}$ and a, b constants that depend on the initial conditions.

The technology of *normal modes*, which we introduce in this section, is a way of formalizing this observation, and applying it systematically to more complicated systems.

§4.1 Canonical kinetic terms

Let us restrict ourselves to the neighbourhood of minima of the potential. Assume, to start with, that we have a Lagrangian

$$L = \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i^2 - V(\mathbf{q})\,. \tag{4.1.1}$$

This particularly simple form for the kinetic term $T = \frac{1}{2}\sum_{i=1}^{n}\dot{q}_i^2$ is known as a *canonical* kinetic term.

Assume that there is a stationary point of $V(\mathbf{q})$ at $\mathbf{q} = 0$, that is

$$\frac{\partial V}{\partial q_i}\Big|_{\mathbf{q}=0} = 0 \quad \forall i\,.$$

If the stationary point we are interested in is at some other position $\mathbf{q} = (a_1, \ldots, a_N)$, we can simply introduce new variables $q_i' = q_i - a_i$ such that the stationary point is now at $\mathbf{q}' = 0$. Clearly in doing this the form of equation (4.1.1) is preserved, so for simplicity we will assume henceforth that the stationary point we are studying is indeed at $\mathbf{q} = 0$.

We can write an approximate Lagrangian, describing the dynamics around this extremum, by expanding $V(\mathbf{q})$ to second order in $\mathbf{q}$

$$L_{\text{approx}} = \frac{1}{2}\sum_{i=1}^{n} \dot{q}_i^2 - V(\mathbf{0}) - \frac{1}{2}\sum_{i,j} \mathsf{A}_{ij} q_i q_j \tag{4.1.2}$$

with

$$\mathsf{A}_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q=0} . \tag{4.1.3}$$

The equations of motion arising from the approximate Lagrangian are given in matrix notation by

$$\ddot{\mathbf{q}} + \mathsf{A}\mathbf{q} = 0 \,. \tag{4.1.4}$$

Note that the $V(\mathbf{0})$ term does not affect the equations of motion. The approximate equations of motion are linear, since they can be written as

$$D_{\mathsf{A}}\mathbf{q} \coloneqq \left(\frac{d^2}{dt^2} + \mathsf{A}\right)\mathbf{q} = 0 \,. \tag{4.1.5}$$

where we have defined $D_{\mathsf{A}} \coloneqq \frac{d^2}{dt^2} + \mathsf{A}$. This is a linear operator, meaning that given any two vectors $\mathbf{a}$ and $\mathbf{b}$ we have $D_{\mathsf{A}}(\mathbf{a}+\mathbf{b}) = D_{\mathsf{A}}\mathbf{a} + D_{\mathsf{A}}\mathbf{b}$, and also for any $c \in \mathbb{R}$ and vector $\mathbf{a}$ we have $D_{\mathsf{A}}(c\mathbf{a}) = cD_{\mathsf{A}}\mathbf{a}$. We have n equations, and the equations are of second order and linear, so we expect to be able to express any solution of the approximate equations of motion as a linear superposition of some $2n$ basic solutions.

To find these solutions, let us start by noticing that the $n \times n$ matrix A is real and symmetric (for any potential whose second partial derivatives are continuous, which will be the case during this course), so it has real eigenvalues and eigenvectors. We denote the set of eigenvalues of A by $\lambda^{(i)}$, and the n corresponding eigenvectors by $\mathbf{v}^{(i)}$, so that

$$\mathsf{A}\mathbf{v}^{(i)} = \lambda^{(i)}\mathbf{v}^{(i)} \,. \tag{4.1.6}$$

Let us now take an ansatz[11]

$$\mathbf{q}^{(i)}(t) = f^{(i)}(t)\mathbf{v}^{(i)} \tag{4.1.7}$$

for some function $f^{(i)}(t)$ that we will determine. Since the eigenvector $\mathbf{v}^{(i)}$ has eigenvalue $\lambda^{(i)}$ we have

$$\begin{aligned}\left(\frac{d^2}{dt^2} + \mathsf{A}\right)\mathbf{q}^{(i)}(t) &= \left(\frac{d^2}{dt^2} + \mathsf{A}\right) f^{(i)}(t)\mathbf{v}^{(i)} \\ &= \mathbf{v}^{(i)}\left(\frac{d^2}{dt^2} + \lambda^{(i)}\right) f^{(i)}(t) \\ &= 0 \,.\end{aligned}$$

[11] An *ansatz* is an assumed form for the solution of the problem. We test the assumption by inserting the ansatz into the equation, and verifying that it does provide a solution for an appropriate choice of $f(t)$.

Since $\mathbf{v}^{(i)} \neq 0$, this implies that

$$\left(\frac{d^2}{dt^2} + \lambda^{(i)}\right) f^{(i)}(t) = 0\,.$$

The form of the solution depends on the sign of $\lambda^{(i)}$:

$$f^{(i)}(t) = \begin{cases} \alpha^{(i)} \cos(\sqrt{\lambda^{(i)}}t) + \beta^{(i)} \sin(\sqrt{\lambda^{(i)}}t) & \text{if } \lambda^{(i)} > 0 \\ C^{(i)}t + D^{(i)} & \text{if } \lambda^{(i)} = 0 \\ \alpha^{(i)} \cosh(\sqrt{-\lambda^{(i)}}t) + \beta^{(i)} \sinh(\sqrt{-\lambda^{(i)}}t) & \text{if } \lambda^{(i)} < 0 \end{cases}$$

where the $\alpha^{(i)}$, $\beta^{(i)}$, $C^{(i)}$ and $D^{(i)}$ are constants to be fixed by initial conditions. Note that whatever the value of $\lambda^{(i)}$, each eigenvector leads to a two-dimensional space of solutions. Since the eigenvectors span n-dimensional space, our ansatz gives us the full $2n$-dimensional space of solutions to the linear equation. So we can write the general solution of the system in terms of the ansatz (4.1.7) as

$$\mathbf{q}(t) = \sum_{i=1}^{N} \mathbf{v}^{(i)} f^{(i)}(t)\,.$$

The qualitative behaviour of the solution depends on the sign of the eigenvalues $\lambda^{(i)}$. For $\lambda^{(i)}$ all being positive we are at a local minimum, and we have oscillatory behaviour around the minimum. If we have a negative eigenvalue we instead have exponential behaviour away from the stationary point. This agrees with expectations: if we are at a maximum along some direction, small perturbations away the point will quickly grow, and we are trying to expand around an unstable solution. Finally, zero eigenvalues are associated with motion with constant velocity, displaying no oscillatory behaviour.

Definition 4.1.1. Each basic solution

$$\mathbf{q}(t) = \mathbf{v}^{(i)} \left(\alpha^{(i)} \cos(\sqrt{\lambda^{(i)}}t) + \beta^{(i)} \sin(\sqrt{\lambda^{(i)}}t)\right)$$

associated with an eigenvalue $\lambda^{(i)} > 0$ is a *normal mode*.

Definition 4.1.2. Each basic solution

$$\mathbf{q}(t) = \mathbf{v}^{(i)} \left(C^{(i)}t + D^{(i)}\right)$$

associated with a zero eigenvalue $\lambda^{(i)} = 0$ is a *zero mode*.

Definition 4.1.3. Each basic solution

$$\mathbf{q}(t) = \mathbf{v}^{(i)} \left(\alpha^{(i)} \cosh(\sqrt{-\lambda^{(i)}}t) + \beta^{(i)} \sinh(\sqrt{-\lambda^{(i)}}t)\right)$$

associated with an eigenvalue $\lambda^{(i)} < 0$ is an *instability*.

The general solution in the absence of instabilities is the superposition of the ordinary normal modes for the non-zero eigenvalues and the zero modes

$$\mathbf{q}(t) = \sum_{\substack{i=1 \\ \lambda^{(i)}\neq 0}}^{n} \mathbf{v}^{(i)} \left(\alpha^{(i)} \cos(\omega^{(i)} t + \beta^{(i)} \sin(\omega^{(i)} t))\right) + \sum_{\substack{i=1 \\ \lambda^{(i)}=0}}^{n} \mathbf{v}^{(i)} \left(C^{(i)} t + D^{(i)}\right) .$$

Note 4.1.4

Let me emphasize that the existence of zero modes is fairly brittle: if we slightly deform our starting potential $V(\mathbf{q})$ in a generic way, then the eigenvalues of A will generically change slightly, and the zero eigenvalues will generically becomes either positive or negative. So whenever we find a zero mode in a real physical system this tells us very valuable information: we expect to be able to find some principle that restricts the possible deformations of $V(\mathbf{q})$!

As an example, imagine that we have two particles with the same mass moving in one dimension, located at x_1 and x_2. Assume that the physics is independent of the choice of origin of coordinates, or equivalently that there is a symmetry

$$x_1 \to x_1 + \epsilon a$$
$$x_2 \to x_2 + \epsilon a$$

for any constant a. Then the potential can only depend on the difference $x_1 - x_2$:

$$L = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) - V(x_1 - x_2) .$$

This symmetry will then always lead to the existence of a zero mode, associated with translation of the centre of mass of the system. We can see this explicitly if we introduce new coordinates $x_+ \coloneqq \frac{1}{\sqrt{2}}(x_1 + x_2)$, $x_- \coloneqq \frac{1}{\sqrt{2}}(x_1 - x_2)$. Our Lagrangian becomes

$$L = \frac{1}{2} m(\dot{x}_+^2 + \dot{x}_-^2) - V(\sqrt{2} x_-)$$

which clearly leads to a zero mode for x_+, no matter the specific form of V. So in this case we find that the existence of the zero mode is ultimately protected by the translation symmetry!