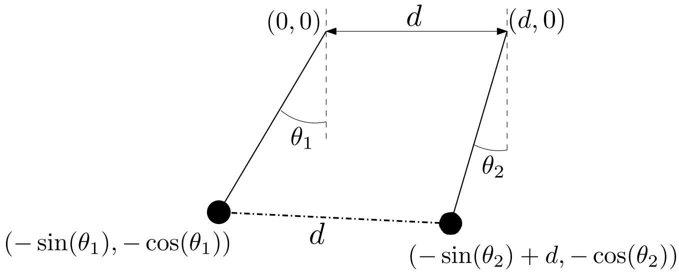
Example 4.1.5. *Consider two pendula, each of length one with mass one, suspended a distance d apart. Connecting the masses is a spring of constant κ and also of natural length d.*



The velocity of the left hand mass is simply $((-\cos(\theta_1)\dot{\theta})_1^2 + (\sin(\theta_1)\dot{\theta}_1)^2) = \dot{\theta}_1^2$. *We get a similar result for the right hand mass so the total kinetic energy T is*

$$T = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right).$$

The potential comes from gravity, which gives a contribution $g(-\cos(\theta_1) - \cos(\theta_2))$, *and from the spring. For a spring of constant κ, its potential energy is given by $\kappa(l-d)^2/2$, where $l-d$ is the extension of the spring. The length l of the spring is given by Pythagoras Theorem as*

$$l = \sqrt{(\sin(\theta_1) - \sin(\theta_2) + d)^2 + (\cos(\theta_1) - \cos(\theta_2))^2}.$$

Thus the Lagrangian for the system is given by

$$\begin{aligned} L &= \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) + g(\cos(\theta_1) + \cos(\theta_2)) \\ &- \frac{\kappa}{2}\left(\sqrt{(\sin(\theta_1) - \sin(\theta_2) + d)^2 + (\cos(\theta_1) - \cos(\theta_2))^2} - d\right)^2. \end{aligned}$$

Finding the exact solution to the equations of motion resulting from this Lagrangian seems hopeless. However, it is clear that the system would be happy to sit at $\theta_1 = \theta_2 = 0$, as this configuration minimises both the gravitational potential energy, and the spring energy since the spring would be at its natural unextended length d. Let us now try to find an approximate Lagrangian which describes the system when $\theta_i \ll 1$.

Approximating the gravitational potential is easy. $\cos(\theta) \approx 1 - \theta^2/2 + O(\theta^4)$ *so we can take*

$$-g(\cos(\theta_1) + \cos(\theta_2)) = -g\left(2 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2}\right).$$

The constant term $-2g$ can be discarded using the usual reason that additions of constants to potentials/Lagrangians has no effect. The spring potential looks more tricky to deal with, but note that to calculate $\kappa(l-d)^2/2$ to quadratic order in the small θ_i we only need to calculate $l-d$ to order θ_i, since it is linear in the θ_i:

$$\begin{aligned} l - d &= \sqrt{(\sin(\theta_1) - \sin(\theta_2) + d)^2 + (\cos(\theta_1) - \cos(\theta_2))^2} - d \\ &= \sqrt{(\sin(\theta_1) - \sin(\theta_2) + d)^2} - d + O(\theta^2) \\ &= \theta_1 - \theta_2. \end{aligned}$$

Finally we can write the approximate Lagrangian as

$$L_{approx} = \frac{1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{g}{2}\left(\theta_1^2 + \theta_2^2\right) - \frac{\kappa}{2}\left(\theta_1 - \theta_2\right)^2.$$

The equations which follow from this are

$$\begin{aligned} \ddot{\theta}_1 + (g+\kappa)\theta_1 - \kappa\theta_2 &= 0 \\ \ddot{\theta}_2 - \kappa\theta_1 + (g+\kappa)\theta_2 &= 0. \end{aligned}$$

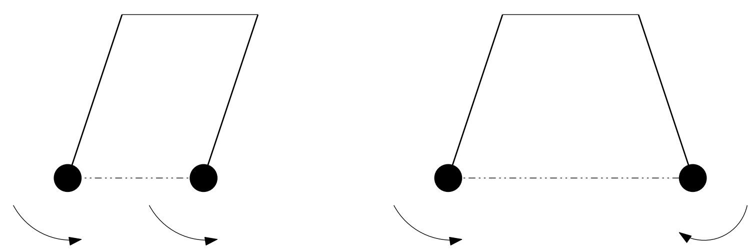
If one arranges the equations of motion in this way, so that all the terms proportional to θ_1 *and those proportional to* θ_2 *appear in columns then it is straightforward to read the elements of matrix* A *from the equations as*

$$A = \begin{pmatrix} g+\kappa & -\kappa \\ -\kappa & g+\kappa \end{pmatrix}.$$

Solving for the eigenvalues of A *we find that* $\lambda = g$ *or* $g + 2\kappa$*, with eigenvectors* $(1,1)$ *or* $(1,-1)$ *respectively. So we can write the normal modes as*

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\pm i\sqrt{g}t} \quad \text{or} \quad \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\pm i\sqrt{g+2\kappa}t}$$

The first of these has $\theta_1 = \theta_2$ *whilst the second has* $\theta_1 = -\theta_2$*. These two normal modes can be pictured as follows: For the normal mode which has* $\theta_1 = \theta_2$*, the spring always*



remains exactly length d *and therefore remains unextended and exerts no force. The result of this is that the angular frequency or this normal mode is* $\sqrt{g}$ *which does not involve* κ *the spring constant. On the other hand, for the second normal mode the pendula move in opposite directions, and in this case the spring stretches and contracts, enhancing the effect of gravity which results in an angular frequency* $\sqrt{g+2\kappa}$ *which is greater than that of the first normal mode in which only gravity plays a role.*

The general solution of the system is thus given by

$$\begin{aligned} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left[\alpha^{(1)} \cos(t\sqrt{g}) + \beta^{(1)} \sin(t\sqrt{g})\right] \\ &\quad + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left[\alpha^{(2)} \cos(t\sqrt{g+2\kappa}) + \beta^{(2)} \sin(t\sqrt{g+2\kappa})\right] \end{aligned}$$

with $\alpha^{(i)}$ *and* $\beta^{(i)}$ *arbitrary constants. To see how the general solution we found helps in practice when studying the motion of the system, let us use this solution to study what happens if we release the two masses from rest at* $t = 0$ *from* $\theta_1 = -\theta_2 = \delta$. *Setting* $\theta_1 = -\theta_2 = \delta$ *at* $t = 0$ *we find*

$$\begin{pmatrix} \delta \\ -\delta \end{pmatrix} = \begin{pmatrix} \alpha^{(1)} + \alpha^{(2)} \\ \alpha^{(1)} - \alpha^{(2)} \end{pmatrix} \tag{4.1.8}$$

so $\alpha^{(1)} = 0$ *and* $\alpha^{(2)} = \delta$. *Similarly, the condition that the masses are released from rest is encoded in*

$$\begin{pmatrix} \dot\theta_1(t=0) \\ \dot\theta_2(t=0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.1.9}$$

which taking derivatives in our general solution is easily shown to lead to

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \beta^{(1)} + \beta^{(2)} \\ \beta^{(1)} - \beta^{(2)} \end{pmatrix} \tag{4.1.10}$$

which implies $\beta^{(1)} = \beta^{(2)} = 0$. *So we find that the motion is given by*

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \delta \\ -\delta \end{pmatrix} \cos(t\sqrt{g + 2\kappa})$$

which is an oscillatory motion in which the masses move oppositely, without changing the centre of mass, as one might have guessed.