4.2 NON-CANONICAL KINETIC TERMS

§4.2 Non-canonical kinetic terms

Finally, we consider configurations with non-canonical kinetic terms of the form

$$L = \frac{1}{2} \sum_{i,j} \mathsf{B}_{ij}(q) \dot{q}_i \dot{q}_j - V(q) \,. \tag{4.2.1}$$

We still obtain a linear differential operator if we restrict $B(q) \rightarrow B(0)$. Physically, this corresponds to considering oscillations with not too much kinetic energy, which makes sense if we want to stay at the minimum. The resulting equations of motion are

$$\mathsf{B}\ddot{\mathbf{q}} + \mathsf{A}\mathbf{q} = 0 \tag{4.2.2}$$

where we have defined $B \equiv B(0)$, a constant matrix. B generally does not have zero eigenvalues (since this would correspond to generalised coordinates without a kinetic term), so we assume no zero eigenvalues. This implies that $\det(B) \neq 0$, and so B^{-1} exists. We then have an equivalent set of equations

$$\ddot{\mathbf{q}} + \mathsf{B}^{-1}\mathsf{A}\mathbf{q} = 0 \tag{4.2.3}$$

which reduces to the case we have already studied if you define $C := B^{-1}A$.

There is one small subtlety that needs to be mentioned here: the fact that A was symmetric was quite important in our discussion above, since it ensured that its eigenvalues were real, but in general $B^{-1}A$ will not be symmetric, even if both B^{-1} and A separately are. Let us assume that A is positive semi-definite: that is, all its eigenvalues are either positive or zero. Then it exists a symmetric matrix $A^{\frac{1}{2}}$ such that $(A^{\frac{1}{2}})^2 = A$.¹² We can use this matrix to rewrite

$$\mathsf{C} := \mathsf{B}^{-1}\mathsf{A} = \mathsf{A}^{-\frac{1}{2}} \left(\mathsf{A}^{\frac{1}{2}}\mathsf{B}^{-1}\mathsf{A}^{\frac{1}{2}} \right) \mathsf{A}^{\frac{1}{2}}$$

so we find that C is similar (in the sense of similarity transformations of matrices) to $A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$. This matrix is manifestly symmetric, so its eigenvalues are real. Since similar matrices have the same eigenvalues, the eigenvalues of C will be real too. It is straightforward to check that if v is an eigenvector of $A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ with eigenvalue λ , then $A^{-\frac{1}{2}}v$ will be an eigenvector of C with the same eigenvalue. So, in practice, we can simply compute the eigenvalues and eigenvectors of C, and proceed as we did above.

¹²The simplest way to prove this is to note that A is real symmetric, and thus diagonalisable by an orthogonal transformation O as $A = ODO^t$. We can then define $A^{\frac{1}{2}} = OD^{\frac{1}{2}}O^t$.