Alternative proof. Here I will present a less straightforward but (in my opinion) more illuminating proof, closer in spirit to the one we used in proving Noether's theorem.

> *This alternative proof is* ***not examinable****.*

Imagine that we take a path $\mathbf{q}(t)$ satisfying the equations of motion, and we displace it to a new path $\mathbf{q}'(t) = \mathbf{q}(t-\epsilon)$. That is, we move the whole path slightly forward in time, keeping its shape. We have

$$\begin{aligned} S' &= \int_{t_0}^{t_1} dt\, L(q_1'(t), \ldots, q_N'(t), \dot{q}_1'(t), \ldots, \dot{q}_N'(t), t) \\ &= \int_{t_0}^{t_1} dt\, L(q_1(t-\epsilon), \ldots, q_N(t-\epsilon), \dot{q}_1(t-\epsilon), \ldots, \dot{q}_N(t-\epsilon), t)\,. \end{aligned}$$

We can compute this expression in two different ways. First, by the Chain Rule, we have that

$$\begin{aligned} &L(q_1(t-\epsilon), \ldots, q_N(t-\epsilon), \dot{q}_1(t-\epsilon), \ldots, \dot{q}_N(t-\epsilon), t) = \\ &\qquad L(q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots, \dot{q}_N(t), t) - \epsilon \left(\sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \right) + \mathcal{O}(\epsilon^2)\,. \end{aligned}$$

Using the Euler-Lagrange equations of motion, we can write this as

$$\begin{aligned} \sum_{i=1}^{N} \frac{\partial L}{\partial q_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i &= \sum_{i=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \\ &= \frac{d}{dt}\left(\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right)\,. \end{aligned}$$

Substituting these expressions into the action, we have just proven that

$$S' = S - \epsilon \left[\sum_{i=1}^{N} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right]_{t_0}^{t_1} + \mathcal{O}(\epsilon^2)\,.$$

On the other hand, introducing a new variable $t' = t - \epsilon$, we can write

$$\begin{aligned} S' &= \int_{t_0}^{t_1} dt\, L(q_1(t-\epsilon), \ldots, q_N(t-\epsilon), \dot{q}_1(t-\epsilon), \ldots, \dot{q}_N(t-\epsilon), t) \\ &= \int_{t_0-\epsilon}^{t_1-\epsilon} dt'\, L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)\,. \end{aligned}$$

We can expand this as a series in ϵ using Leibniz's rule (see equation (A.0.1) in the appendix for a reminder), to get:

$$\begin{aligned} S' &= S + \epsilon \frac{dS'}{d\epsilon}\bigg|_{\epsilon=0} + \mathcal{O}(\epsilon^2) \\ &= S - \epsilon L(q_1(t_1), \ldots, q_N(t_1), \dot{q}_1(t_1), \ldots, \dot{q}_N(t_1), t_1) \\ &\quad + \epsilon L(q_1(t_0), \ldots, q_N(t_0), \dot{q}_1(t_0), \ldots, \dot{q}_N(t_0), t_0) \\ &\quad + \epsilon \left[\int_{t_0-\epsilon}^{t_1-\epsilon} dt' \, \frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial \epsilon} \right]_{\epsilon=0} \\ &\quad + \mathcal{O}(\epsilon^2) \end{aligned}$$

Now we note that that, by the Chain Rule, we have

$$\frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial \epsilon} = \frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial t'}$$

so

$$\left[\int_{t_0-\epsilon}^{t_1-\epsilon} dt' \, \frac{\partial L(q_1(t'), \ldots, q_N(t'), \dot{q}_1(t'), \ldots, \dot{q}_N(t'), t'+\epsilon)}{\partial \epsilon} \right]_{\epsilon=0} = \int_{t_0}^{t_1} dt \, \frac{\partial L(q_1(t), \ldots, q_N(t), \dot{q}_1(t), \ldots, \dot{q}_N(t), t)}{\partial t} \, .$$

The theorem now follows from equating the two expressions for S' that we found. □