Note 4.2.4

It is not obvious that the quantity E that is conserved if $\frac{\partial L}{\partial t} = 0$ is what is usually known as "energy" in classical mechanics. But this is easy to verify. Assume that we have a particle with Lagrangian

$$L = T(\dot{x}_1, \ldots, \dot{x}_N) - V(x_1, \ldots, x_N)$$

with $T(\dot{x}_1, \ldots, \dot{x}_N) = \frac{1}{2}m(\dot{x}_1^2 + \ldots + \dot{x}_d^2)$, as we often do in classical mechanics. Then applying the definition 4.2.1 above one easily finds the expected relation

$$E = T + V .$$

The result holds more generally. Consider a Lagrangian of the form

$$L = \underbrace{\left(\sum_{i,j=1}^{N} K_{ij}(q_1, \ldots, q_N)\dot{q}_i\dot{q}_j \right)}_{T(q,\dot{q})} - V(q)$$

with the $K_{ij}(q)$ and $V(q)$ arbitrary functions on configuration space $\mathcal{C}$. Then it is easy to verify that

$$E = T + V .$$

Example 4.2.5. *Say that we have a spring that becomes weaker with time, with a spring constant $\kappa(t) = e^{-t}$. A mass attached to the spring can then be described by a Lagrangian*

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\kappa(t)x^2 .$$

The resulting equation of motion is

$$m\ddot{x} + \kappa(t)x = 0 .$$

The energy of the system is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\kappa(t)x^2 .$$

Since the Lagrangian depends explicitly on time, we expect that energy is not conserved. And indeed:

$$\begin{aligned} \frac{dE}{dt} &= m\dot{x}\ddot{x} + \kappa(t)x\dot{x} + \frac{1}{2}x^2\frac{d\kappa(t)}{dt} \\ &= \dot{x}(m\ddot{x} + \kappa(t)x) + \frac{1}{2}x^2\frac{d\kappa(t)}{dt} \\ &= \frac{1}{2}x^2\frac{d\kappa(t)}{dt} \end{aligned}$$

where in the last step we have used the equation of motion. On the other hand

$$\frac{\partial L}{\partial t} = -\frac{1}{2}x^2\frac{d\kappa(t)}{dt}$$

since time appears explicitly only in $\kappa(t)$. So we have verified that

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}\,.$$

Example 4.2.6. *Note that our definition 4.2.1 for the energy does not require the Lagrangian to have the specific form $L = T - V$. Consider for instance the Lagrangian*

$$L = -mc\left(\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}\right)\,.$$

(This specific Lagrangian is in fact fairly important, as it describes the motion of a particle in $\mathbb{R}^3$ in special relativity, where m is the mass of the particle and c is the speed of light.) Definition 4.2.1 gives

$$E = \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}} - L\,.$$

We have

$$\dot{x}\frac{\partial L}{\partial \dot{x}} = \frac{mc\dot{x}^2}{\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}},$$

and similarly for $\dot{y}$ and $\dot{z}$. Putting everything together we find

$$\begin{aligned} E &= \frac{mc(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}{\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}} + mc\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2} \\ &= \frac{mc^3}{\sqrt{c^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}} \\ &= \frac{mc^2}{\sqrt{1 - \left(\frac{\dot{\mathbf{x}}}{c}\right)^2}}\,, \end{aligned}$$

where $\dot{\mathbf{x}} \coloneqq \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. This result is often written as $E = mc^2\gamma$, with $\gamma \coloneqq 1/\sqrt{1 - (\dot{\mathbf{x}}/c)^2}$ known as the "Lorentz factor", which for velocities small relative to the speed of light is approximately $1 + \frac{1}{2}(\dot{\mathbf{x}}/c)^2$, giving $E = mc^2 + \frac{1}{2}m\dot{\mathbf{x}}^2 + \ldots$.