§5 Fields and The Wave Equation

§5.1 Variational Principle for Continuous Systems

In the next section we will derive the equations of motion for the string. Before going into the details of that particular system, we will derive in general how to deduce the Euler-Lagrange equations for fields, which is a simple generalisation of what we did in the case of systems with a finite number of degrees of freedom.

Assume that we can express the action S in terms of some *Lagrangian density* $\mathcal{L}$ (we will determine $\mathcal{L}$ for the string in the next section)

$$S = \int dt \int dx \ \mathcal{L}(u, u_t, u_x, x, t)$$

where we have introduced for convenience the notation

$$u_x \coloneqq \frac{\partial u}{\partial x} \qquad ; \qquad u_t \coloneqq \frac{\partial u}{\partial t} \,.$$

Note 5.1.1

I emphasize that that the "x" coordinate plays a significantly different role in field theory than it did for the point particle: in field theory "x" is a coordinate that fields depend on, and it is on the same footing as "t". They are, in particular, independent variables, and they are not generalized coordinates.

On the other hand, for the point particle we often denoted by "$x(t)$" the position of the particle, which was a generalized coordinate that for any given path was a function of time. In field theory the closest thing to this "x" is the field value "$u(x, t)$".

Note 5.1.2

There is an important notational point that I want to clarify: say that we have a Lagrangian density $\mathcal{L}(u, u_x, u_t, x, t)$ depending on the field, its first derivatives, and x and t themselves. Then we have two notions of "derivative of $\mathcal{L}$ with respect to t" (the following discussion generalizes straightforwardly to x, so I will not consider this case separately). We might mean either:

1. The derivative with respect to any explicit appearances of t, keeping u, u_x, u_t and x fixed.

2. The derivative of $\mathcal{L}$ with respect to t, taking into account that u, u_x and u_t are

functions of t, so we need to use the chain rule.

In the context of the point particle, we denoted the first derivative "$\partial/\partial t$" and the second "d/dt".

In the context of field theory it is more common and useful to switch conventions, and denote the second option by $\partial\mathcal{L}/\partial t$. That is, we define:

$$\begin{aligned}\frac{\partial\mathcal{L}(u,u_x,u_t,x,t)}{\partial t} := \lim_{h\to 0}\frac{1}{h}(&\mathcal{L}(u(x,t+h),u_x(x,t+h),u_t(x,t+h),x,t+h)\\ &-\mathcal{L}(u(x,t),u_x(x,t),u_t(x,t),x,t))\end{aligned}$$

We will simply never need to consider the first notion of partial derivative in the context of fields during this course, so this leads to no ambiguity.

The main reason to switch conventions is that this reproduces the natural definition:

$$u_t := \frac{\partial u(x,t)}{\partial t}$$

that we gave above, since the meaning of the derivative here is the usual one: we are varying t keeping x fixed.

In this case we expect to be able to derive the equations of motion for the system by making use of the variational principle we discussed in previous sections. To see how this goes, consider a solution of the equations of motion $u_s(x,t)$, and consider a small variation $\delta u(x,t)$ around it:

$$u(x,t) = u_s(x,t) + \delta u(x,t)\,.$$

If u_s is indeed an stationary function for the action, we expect the first order change in S to vanish:

$$\delta S = S[u_s+\delta u] - S[u_s] = \int dt\int dx\,\left(\delta u\frac{\partial\mathcal{L}}{\partial u} + \delta u_x\frac{\partial\mathcal{L}}{\partial u_x} + \delta u_t\frac{\partial\mathcal{L}}{\partial u_t}\right) + O((\delta u)^2)$$

We will work to first order, and drop the $O((\delta u)^2)$ terms henceforth. Now, for our variations we have

$$\delta u_x = \delta\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(\delta u) \qquad ; \qquad \delta u_t = \delta\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}(\delta u)$$

which allows us to integrate δS by parts, in order to obtain

$$\delta S = \int dt\int dx\;\delta u\left(\frac{\partial\mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial u_t}\right)\right) + \int dt\left[\delta u\frac{\partial\mathcal{L}}{\partial u_x}\right]_{x_i}^{x_f} + \int dx\left[\delta u\frac{\partial\mathcal{L}}{\partial u_t}\right]_{t_i}^{t_f}$$

If we assume that we hold u fixed at the endpoints both in x and t the last two terms on the right cancel. Imposing $\delta S = 0$ for arbitrary δu then implies, by the fundamental lemma of the calculus of variations,[13] that the generalised Euler-Lagrange equations for fields

$$\boxed{\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) = 0\,.}$$

Here are some easy generalisations. Clearly, if we have n fields $u^{(i)}$ we end up with n generalised equations of motion:

$$\frac{\partial \mathcal{L}}{\partial u^{(i)}} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x^{(i)}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t^{(i)}}\right) = 0 \qquad \text{for all } i\,.$$

Another possible easy generalisation is considering fields that depend on more coordinates than two. If we replace (t, x) by a set of d coordinates x_i we have

$$\frac{\partial \mathcal{L}}{\partial u^{(i)}} - \sum_{k=1}^{d} \frac{\partial}{\partial x_k}\left(\frac{\partial \mathcal{L}}{\partial u_k^{(i)}}\right) = 0 \qquad \text{for all } i$$

where we have defined $u_k^{(i)} := \frac{\partial u^{(i)}}{\partial x_k}$.

[13]The proof that we gave above for the fundamental lemma of the calculus of variations was for functions of a single variable. I leave it as a small exercise to generalise the proof to the case of functions of multiple variables.