§5.3 D'Alembert's Solution to the Wave Equation

The general solution to the wave equation in one spatial dimension was given by *D'Alembert*, and it is simply

$$u(x,t) = f(x-ct) + g(x+ct)$$

where f and g are arbitrary functions. The part of the solution $f(x-ct)$ corresponds to a wave moving to the right with speed c, whilst the remaining part $g(x+ct)$ corresponds to a wave moving to the left with speed c.

Theorem 5.3.1. *D'Alembert's solution $u(x,t) = f(x-ct)+g(x+ct)$ is the general solution to the wave equation.*

Proof. We introduce new variables $x_+ = x + ct$ and $x_- = x - ct$, or equivalently $x = \frac{1}{2}(x_+ + x_-)$ and $t = \frac{1}{2c}(x_+ - x_-)$. By the Chain Rule:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x_+}\frac{\partial x_+}{\partial x} + \frac{\partial u}{\partial x_-}\frac{\partial x_-}{\partial x} = \frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-}\,,\\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x_+}\frac{\partial x_+}{\partial t} + \frac{\partial u}{\partial x_-}\frac{\partial x_-}{\partial t} = c\left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-}\right)\,.\end{aligned}$$

Taking derivatives again, once more using the Chain Rule:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-}\right) = \frac{\partial}{\partial x_+}\left(\frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-}\right) + \frac{\partial}{\partial x_-}\left(\frac{\partial u}{\partial x_+} + \frac{\partial u}{\partial x_-}\right)\\ &= \frac{\partial^2 u}{\partial x_+^2} + \frac{\partial^2 u}{\partial x_-^2} + 2\frac{\partial^2 u}{\partial x_+\partial x_-}\\ \frac{\partial^2 u}{\partial t^2} &= c\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-}\right) = c^2\frac{\partial}{\partial x_+}\left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-}\right) - \frac{\partial}{\partial x_-}\left(\frac{\partial u}{\partial x_+} - \frac{\partial u}{\partial x_-}\right)\\ &= c^2\left(\frac{\partial^2 u}{\partial x_+^2} + \frac{\partial^2 u}{\partial x_-^2} - 2\frac{\partial^2 u}{\partial x_+\partial x_-}\right)\end{aligned}$$

As usual, we have used the assumption that partial derivatives commute. We see that in these variables the wave equation $u_{tt} = c^2 u_{xx}$ becomes

$$u_{tt} - c^2 u_{xx} = -4c^2\frac{\partial^2 u(x_+, x_-)}{\partial x_+\partial x_-} = 0\,.$$

The general solution of this equation is indeed

$$u(x_+, x_-) = f(x_-) + g(x_+)\,.$$

□

In practice, we are often interested in understanding what happens if we release a string from a given configuration. How does the string evolve? This is an initial value problem, which D'Alembert also solved in general. Assume that we are told that at $t = 0$ the string has profile $\psi(x)$, that is

$$u(x, 0) = \varphi(x)$$

and in addition we know with which speed the string is moving at that instant:

$$u_t(x, 0) = \psi(x)\,.$$

In terms of f and g, which parametrise the general form of the solution, these equation are

$$f(x) + g(x) = \varphi(x)$$

and

$$-cf'(x) + cg'(x) = \psi(x)\,.$$

This last equation can be integrated (formally) to give

$$g(x) - f(x) = d + \frac{1}{c}\int_{-\infty}^{x} ds\, \psi(s)$$

with d some unknown constant. We now have two equations for two unknowns, so solving for f and g we find

$$\begin{aligned} f(x) &= \frac{1}{2}\left(\varphi(x) - d - \frac{1}{c}\int_{-\infty}^{x} ds\, \psi(s)\right) \\ g(x) &= \frac{1}{2}\left(\varphi(x) + d + \frac{1}{c}\int_{-\infty}^{x} ds\, \psi(s)\right) \end{aligned}$$

so we finally find

$$\begin{aligned} u(x, t) &= f(x - ct) + g(x + ct) \\ &= \frac{\varphi(x - ct) + \varphi(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} ds\, \psi(s)\,. \end{aligned}$$