

**Example 5.3.9.** A system whose Lagrangian is given by

$$L = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{r^2}{2}.$$

We define the momenta to be

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = \dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta} \end{aligned}$$

so that

$$\begin{aligned} \dot{r} &= p_r, \\ \dot{\theta} &= \frac{p_\theta}{r^2}. \end{aligned}$$

The Hamiltonian is given by

$$\begin{aligned} H &= p_r \dot{r} + p_\theta \dot{\theta} - \left( \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{r^2}{2} \right) \\ &= p_r^2 + p_\theta \left( \frac{p_\theta}{r^2} \right) - \frac{1}{2} \left( p_r^2 + r^2 \left( \frac{p_\theta}{r^2} \right)^2 - \frac{r^2}{2} \right) \\ &= \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + \frac{r^2}{2}. \end{aligned}$$

Hamilton's Equations of Motion tell us that

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = p_r \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2} \\ \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{r^3} - r \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0. \end{aligned}$$

Note that the first two equations here simply reproduce the results of expressing the  $\mathbf{q}$  in terms of the  $\mathbf{p}$ 's. This is always the case when we derive the Hamiltonian system from a Lagrangian system like above. The last equation shows that  $p_\theta$  is conserved as a result of the Hamiltonian being independent of  $\theta$ . The concept of an ignorable coordinate goes over completely from the Lagrangian picture to the Hamiltonian picture. The real 'meat' of the dynamics is in the remaining equation for  $\dot{p}_r$ . Given that  $p_\theta$  is a constant and that  $p_r = \dot{r}$  it can be read as

$$\ddot{r} = \frac{p_\theta^2}{r^3} - r.$$

**Example 5.3.10.** *Suppose we start instead with a Hamiltonian:*

$$H = \frac{p^2}{2} + xp.$$

*Hamilton's equations are*

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p} = p + x \\ \dot{p} &= -\frac{\partial H}{\partial x} = -p.\end{aligned}$$

*Solving the second equation, we have that  $p = Ae^{-t}$ . Substituting this into the first equation we find*

$$\dot{x} - x = Ae^{-t}.$$

*Multiplying through by the integrating factor we find*

$$\frac{d}{dt}(xe^{-t}) = Ae^{-2t}$$

*which can be integrated to give  $x = Ce^t - Ae^{-t}/2$ .*

**Example 5.3.11.** *The following is a Hamiltonian for the damped harmonic oscillator:*

$$H = \frac{e^{-bt}p^2}{2} + \frac{e^{bt}\omega^2x^2}{2}.$$

*Notice that  $H$  explicitly depends on time; this implies that it is not conserved, as we would expect for the damped harmonic oscillator, whose motion dies away to nothing. Hamilton's equation of motion are*

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p} = e^{-bt}p \\ \dot{p} &= -\frac{\partial H}{\partial x} = -\omega^2e^{bt}x.\end{aligned}$$

*Differentiating the first equation with respect to  $t$ , we see that*

$$\begin{aligned}\ddot{x} &= -be^{-bt}p + e^{-bt}\dot{p} \\ &= -b\dot{x} - \omega^2x,\end{aligned}$$

*which is indeed the equation for a damped harmonic oscillator.*